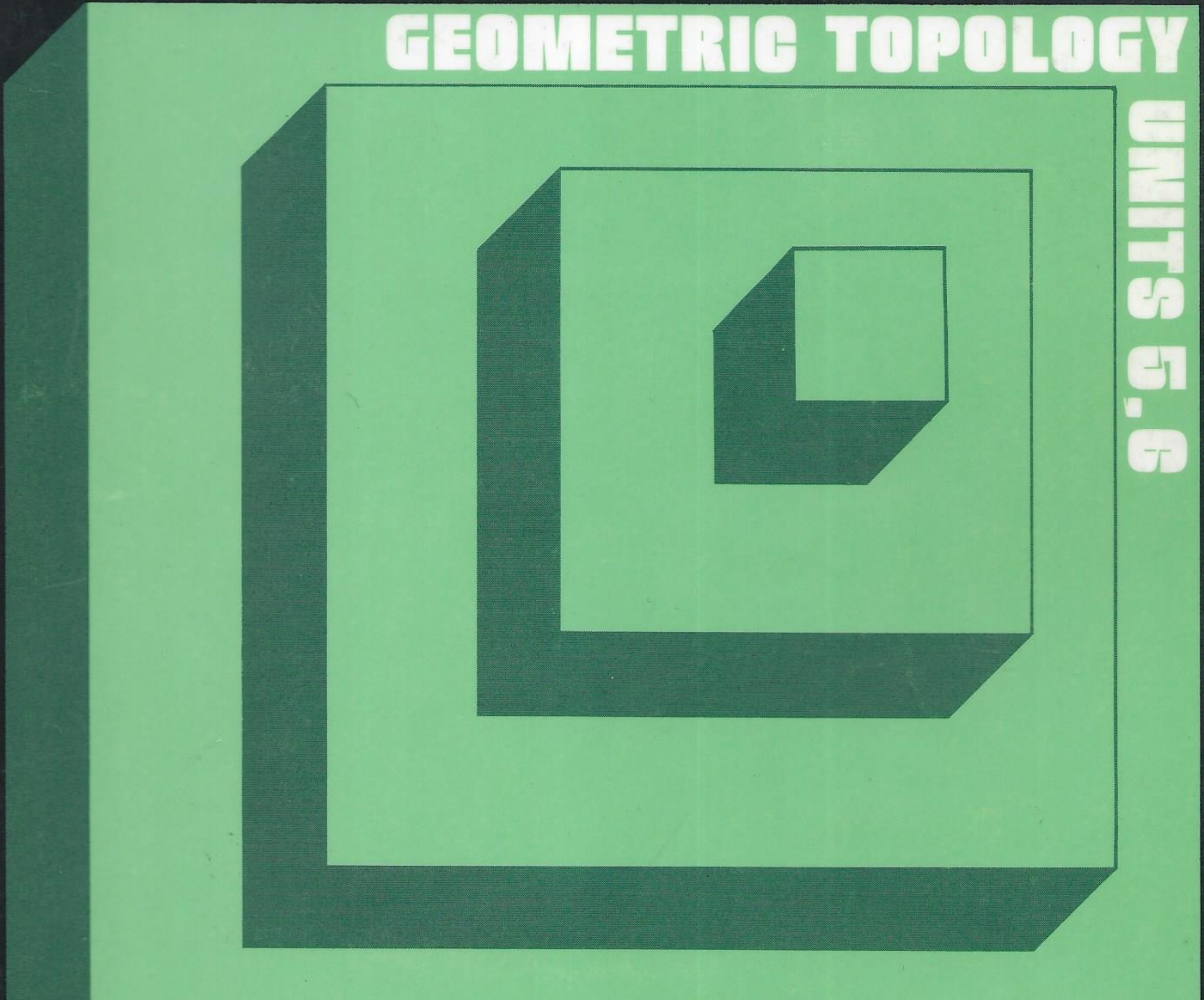
M335 GT 5, 6

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

GEOMETRIC TOPOLOGY

UNITS 5,6



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Prepared by the Course Team

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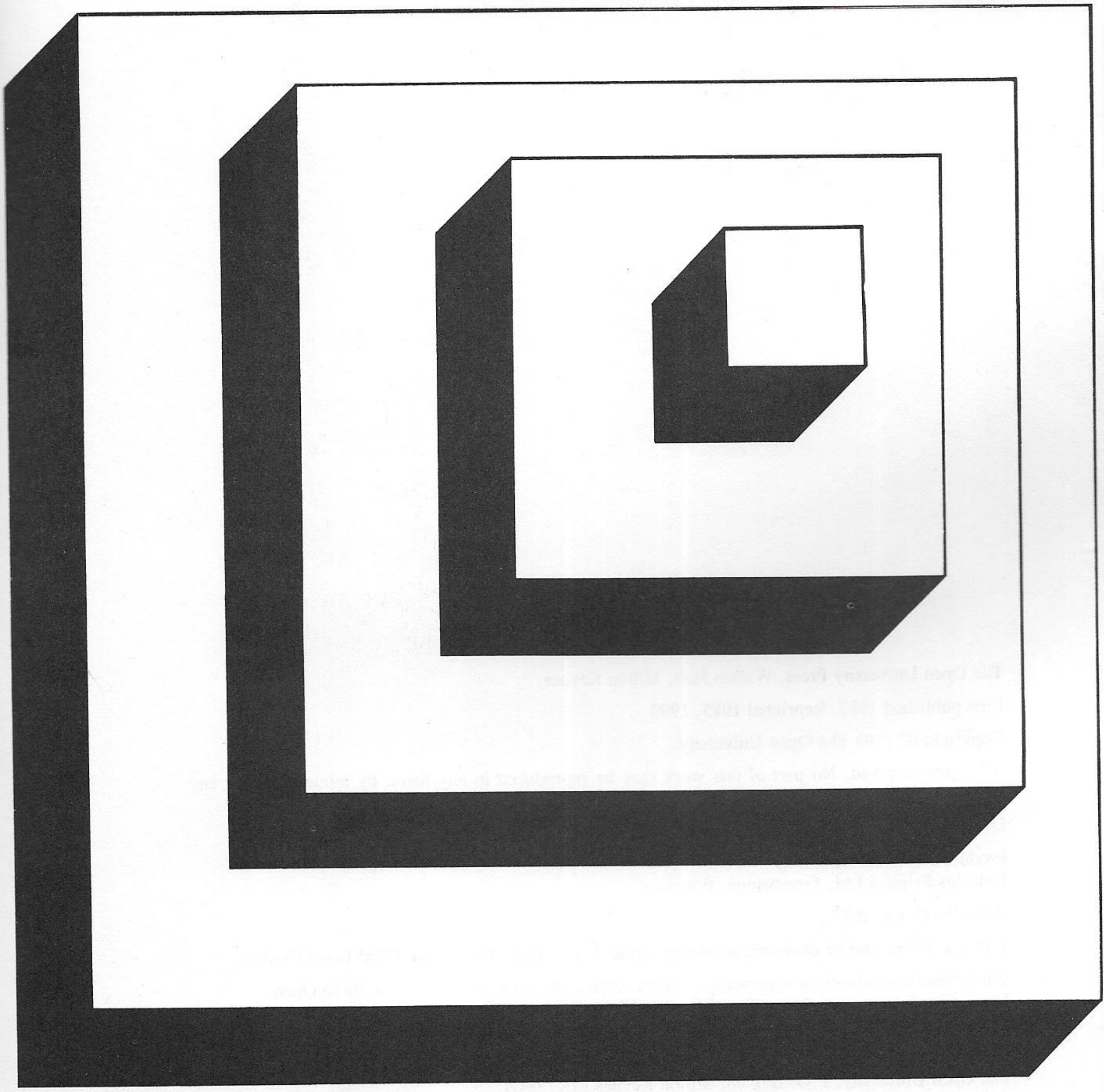

The Open University Press, Walton Hall, Milton Keynes.

First published 1983. Reprinted 1985, 1990

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Designed by the Graphic Design Group of the Open University.

Produced in Great Britain by
Dotesios Printers Ltd, Trowbridge, Wiltshire.

ISBN 0 335 14020 3

This text forms part of the correspondence element of an Open University Third Level Course.

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UNIT 5

COMPLEX LOCI AND THE RIEMANN SPHERE

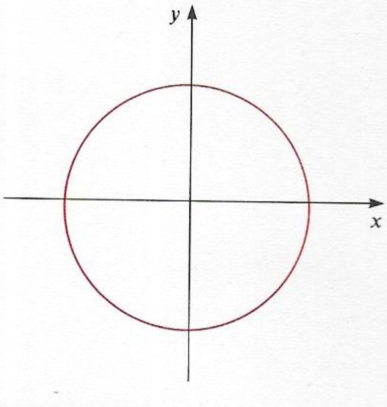
5.1 Complex Loci and the Riemann Sphere

This unit and the next are tied to Chapter 3 of **B**, which discusses two very important and related applications of topology to other branches of mathematics. One application is to the study of complex loci, and the other is to a simple but frequently encountered relation between pairs of spaces. We can describe them both briefly here by an example which we shall treat in more detail later.

Consider the equation $w^2 = z$, where w and z are complex numbers. We can represent the values of z by points in the plane. Points of the locus $w^2 = z$ are pairs (z, w) where, for each non-zero z, there correspond two values of w. In this way, we can think of the locus as a surface spread out over the z-plane, covering it twice. This is because the map from the locus to the z-plane given by projection onto the first coordinate, $(z, w) \longmapsto z$, is particularly simple: every non-zero z-value is the image of exactly two points (z, w) on the locus. This is an illustration of the idea of *covering spaces*; we say that the locus is a two-fold covering of the z-plane. Covering spaces are the second theme of Chapter 3 of **B**. They arise in many areas of mathematics other than the study of loci. In the final section of **B**, the techniques of covering spaces are used to study more complicated loci than quadratics.

Unit 5 is devoted to the topological study of complex quadratic equations. *Unit 6* describes covering spaces and their use in the study of other complex loci.

The aim of this section is to show how we can study geometrical surfaces which are defined by polynomial equations. Recall that an equation in real x and y, such as $x^2 + y^2 = 1$, can be interpreted geometrically, in this case as the circle centre the origin and radius 1. We have in mind a picture like this:



The curve is the graph of the equation.

In this unit we shall look at the situation when x and y are allowed to be *complex* variables. To avoid confusion, we shall use the symbols z and w to denote complex variables, reserving x and y for real quantities.

A complex number z is a pair of real numbers (x, y). Addition of complex numbers is performed coordinate-wise:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Multiplication is defined by the rule

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_2 y_1 + x_1 y_2);$$

for example, $(0, 1) \cdot (0, 1) = (-1, 0)$.

The algebra is easier to remember and use if we introduce a formal symbol, i, as an abbreviation for $(0, 1)$, and write (x, y) as $x + iy$, with the rules:

$$(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2),$$
$$(x_1 + iy) \cdot (x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_2 y_1 + x_1 y_2).$$

The symbol i behaves just like a number, and, as the example above shows, $i \cdot i = -1$. For this reason one sometimes writes $i = \sqrt{-1}$. If you feel you need practice in manipulating complex numbers, consult the Exercise Booklet.

SAQ 1 Suppose $z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Express r and θ in terms of x and y, choosing $0 \leq \theta < 2\pi$. Calculate $re^{i(\theta+\pi)}$, $(re^{i\theta})^2$, $(re^{i(\theta+\pi)})^2$ in terms of r and θ, and deduce that $\rho e^{i\phi}$ has two square roots (which you should write explicitly in terms of ρ and ϕ).

SOLUTION TO SAQ 1 $r^2 = x^2 + y^2, \quad \tan\theta = y/x.$

$$\begin{aligned} re^{i(\theta+\pi)} &= re^{i\theta} \cdot e^{i\pi} \\ &= -re^{i\theta}. \end{aligned}$$

$$(re^{i\theta})^2 = r^2 e^{i\theta} \cdot e^{i\theta} = r^2 e^{2i\theta}.$$

$$\begin{aligned} (re^{i(\theta+\pi)})^2 &= r^2 e^{2i\theta} \cdot e^{2i\pi} \\ &= r^2 e^{2i\theta}. \end{aligned}$$

$$\begin{aligned} \rho e^{i\phi} &= (\sqrt{\rho})^2 e^{2i\phi/2} \\ &= (\sqrt{\rho}\, e^{i\phi/2})^2 \\ &= (\sqrt{\rho}\, e^{i((\phi/2)+\pi)})^2. \quad \blacksquare \end{aligned}$$

If this SAQ caused you no essential difficulty, then you have a sufficient grasp of complex numbers for the purposes of this unit.

Complex numbers can be represented graphically in the real plane, the number $z = x + iy$ being represented by the point (x, y). We shall sometimes call this plane the *plane of complex numbers.* We are interested in what a polynomial equation in z and w represents. Suppose that $f(z, w) = 0$ is a polynomial equation, of degree m in z, and n in w:

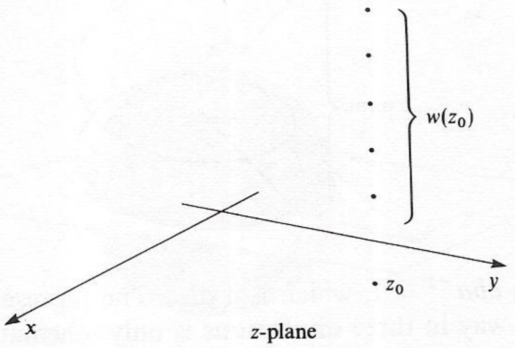
$$f(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) \qquad (*)$$

where each $a_i(z)$ is a polynomial in z of degree no more than m. For each fixed value of z, this is a polynomial in w, which, by the fundamental theorem of algebra, has n roots unless $a_0(z) = 0$. These roots will generally be complex, and may be coincident; coincident roots will interest us greatly later.

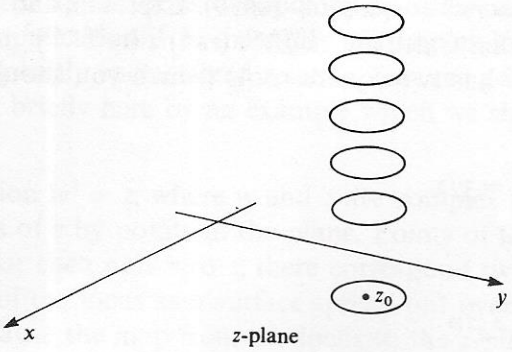
EXAMPLE 1 Suppose $f(z, w) = w^2 + z^2 - 1 = 0$ (so $a_0(z) = 1$, $a_1(z) = 0$, $a_2(z) = z^2 - 1$). If $z^2 - 1 \neq 0$, $w^2 = 1 - z^2$ has two solutions: $w = +\sqrt{z^2 - 1}$, $w = -\sqrt{z^2 - 1}$.

So there are two w-values for each z-value except $z = +1$, $z = -1$. When $z = \pm 1$, w takes only one value (as it happens, zero in each case). $\square$

However, if we try to represent the locus $f(z, w) = w^2 + z^2 - 1 = 0$ graphically, we run into trouble, because we seem to need four real dimensions: two for z and two more for w. We shall see later how to get round this problem when f is quadratic, but first we shall outline the general case $(*)$, which is not significantly harder. Suppose z_0 is such that the n roots of $(*)$ are distinct, i.e. $w(z_0)$ has n values.



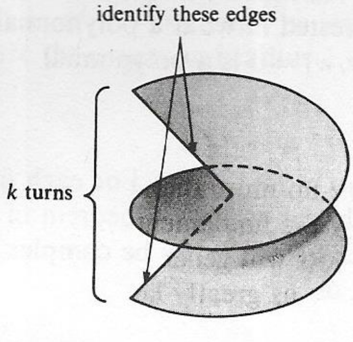
We draw them sitting above z_0. Now let z_0 vary over a small patch in the z-plane. Each of the values of $w(z_0)$ will vary in a continuous way. If we choose a small enough patch around z_0, it is plausible that the values of $w(z_0)$ will remain distinct, and the values will also vary over n small patches.



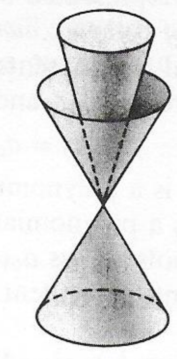
We draw these patches as if they are sitting over the patch containing z_0.

The underlying idea is this: as z varies over the *whole* z-plane, the patches fill out a surface. It is clear that they form something which is (generally) locally disc-like. It is not clear that they form one connected thing, rather than, say, n separate surfaces; and it is not yet clear what happens at points where w takes fewer than n distinct values. It will be the principal object of the rest of this unit to make this clear. We shall see that the separate patches join up over these special values of z in one of two ways:

either like this



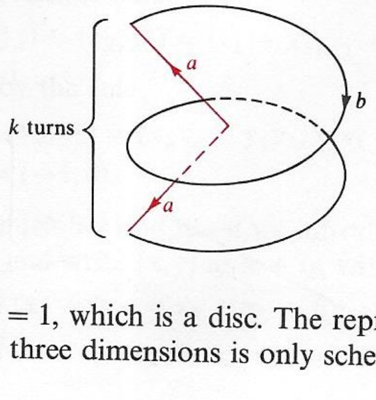
or like this



when (z, w) is called a *branch point*;

when (z, w) is called a *pinch point*.

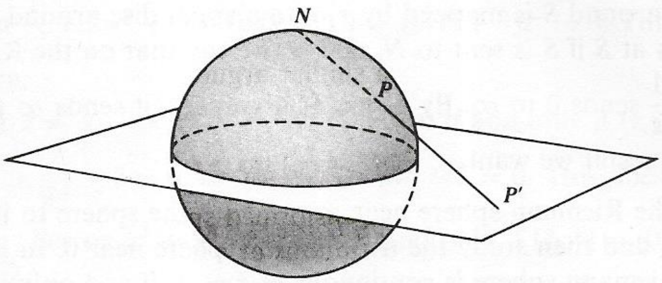
Our intuition lets us down badly when it comes to branch points, because the identifications feel strange. But this is because the graph of $f(z, w) = 0$ requires four real dimensions: two for z and two for w. We shall see later that our algebraic methods are, fortunately, well suited to dealing with this problem. However, it is reasonably clear that a branch point is at the centre of a little disc, so that the sheets of w-values (as they are called) fit together properly at a branch point. We have a figure like this



with edge equation $aba^{-1} = 1$, which is a disc. The representation of (z, w)-values in this way in three dimensions is only schematic. But near a branch point with k turns, one can imagine taking the unit disc in the w-plane, cutting it along a radius, twisting into a spiral shape with k turns, and re-identifying the cut edges. This gives some idea of where the w-values are situated, but the global study of such surfaces is not best attempted by stretching the mind any further in this direction.

On the other hand, a pinch point on a locus means that the locus is *not* a surface, because, as the picture shows, the locus is not locally disc-like at a pinch point.

There is one further problem, which is less difficult to solve: technically the z-plane is not a surface we can study, because it is not compact. This technicality is avoided by stereographic projection. Stereographic projection was studied on pages 9–12 of **B**. It is a map from the unit sphere to the equatorial plane defined by sending P on the sphere to P' in the plane, where P' lies on the line from the north pole (N) to P. There is no image for N under this map.



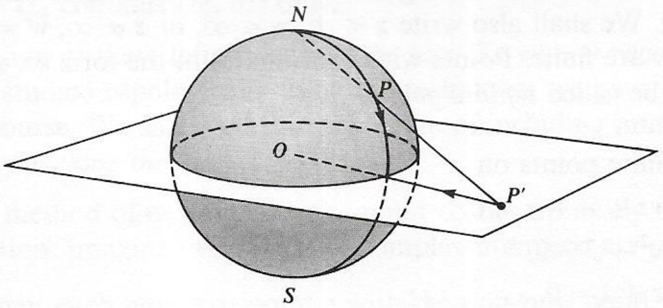
The inverse map from the plane to the sphere associates a unique point, P, on the sphere to each point, P', of the plane. In this way, we can describe each point P of the sphere, except N, by a unique complex number — the complex number which is represented by P'.

SAQ 2 Which complex number represents the south pole, S?

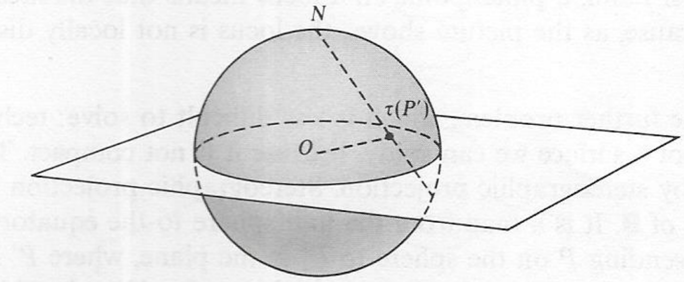
SOLUTION TO SAQ 2 $z = 0$. ■

Stereographic projection can also be thought of as a one-to-one correspondence between the sphere and the plane, with an extra point, called the point at infinity, which corresponds to N. When we do this, we conventionally represent N by the "number" ∞. The sphere, thought of as the plane of complex numbers plus infinity, is then called the *Riemann sphere* after Bernhard Riemann, who successfully pioneered this geometric approach. We shall also use the term "z-sphere".

To study the sphere "near infinity", we need some rules for manipulating ∞ algebraically. Consider the map $\tau: z \longmapsto \dfrac{1}{z}$, which is defined for all $z \neq 0, \infty$. To see how to define $\tau(0)$, consider a ray $P'O$ which passes through the origin of the z-plane.



Under stereographic projection, the image of the ray is a longitude PS passing through the south pole, S. Now consider the image of a point P' under $\tau\colon z \longmapsto \frac{1}{z}$. As P' moves along the ray towards O, so $\tau(P')$ moves along another ray away from O. The stereographic projection of $\tau(P')$ moves along a longitude towards N.



Now concentrate on the Riemann sphere. The above discussion shows that a small disc around S is mapped by τ onto a small disc around N. τ will be continuous at S if S is sent to N, and so we say that on the Riemann sphere, the map $z \longmapsto \frac{1}{z}$ sends 0 to ∞. By a similar argument, it sends ∞ to 0, i.e. N to S. This is the result we want.

To study the Riemann sphere near ∞, we map the sphere to itself by $\tau\colon z \longmapsto 1/z$, and then study the transformed sphere near 0. In particular, a map f from the Riemann sphere is continuous at $z = \infty$ if and only if the map $f \circ \tau$ is continuous at $z = 0$.

Since we have extended the domain of z to include $z = \infty$, we must consider what it means for a point (∞, w) to satisfy an equation $f(z, w) = 0$. We shall also allow w to take the value ∞, so we have three kinds of answer.

(i) A point (∞, w), $w \neq \infty$, satisfies an equation $f(z, w) = 0$ if and only if the point $(\tilde{z}, w) = (0, w)$ satisfies the transformed equation $f\left(\frac{1}{\tilde{z}}, w\right) = 0$.

(ii) A point (z, ∞), $z \neq \infty$, satisfies an equation $f(z, w) = 0$ if and only if the point $(z, \tilde{w}) = (z, 0)$ satisfies the transformed equation $f\left(z, \frac{1}{\tilde{w}}\right) = 0$.

(iii) The point (∞, ∞) satisfies an equation $f(z, w) = 0$ if and only if the point $(\tilde{z}, \tilde{w}) = (0, 0)$ satisfies the transformed equation $f\left(\frac{1}{\tilde{z}}, \frac{1}{\tilde{w}}\right) = 0$.

(In this context, "transformed equation" means substituting $\frac{1}{\tilde{z}}$ for z, or $\frac{1}{\tilde{w}}$ for w, or both, into $f(z, w) = 0$ and then multiplying by a power of $\tilde{z}$, or $\tilde{w}$, or both, to give a polynomial equation in $\tilde{z}, \tilde{w}$. The method will become clear in the SAQs below.)

These investigations are best carried out sequentially, and are illustrated in the following examples. Incidentally, our choice of the "tilde" notation $(\tilde{z}, \tilde{w})$ rather than **B**'s z', w' is not idle — we believe you will make fewer mistakes with tildes since you do not have to distinguish z' from z^2, which are easily confused in handwriting. We shall also write $z < \infty$, $w < \infty$, or $z \neq \infty$, $w \neq \infty$, to indicate that z and w are finite. Points with coordinates of the form (z, ∞), (∞, w), (∞, ∞) will be called *infinite points.*

SAQ 3 Find the infinite points on

(a) $z^2 + 2wz + w + z = 0$,

(b) $z^2 + 2w^2 + 3 = 0$,

(c) $4wz + z + 1 = 0$.

SOLUTION TO SAQ 3

(a) (i) (∞, w), $w \neq \infty$, lies on $z^2 + 2wz + w + z = 0$ if and only if $(\tilde{z}, w) = (0, w)$ lies on $\frac{1}{\tilde{z}^2} + \frac{2w}{\tilde{z}} + w + \frac{1}{\tilde{z}} = 0$, i.e. on $1 + 2w\tilde{z} + w\tilde{z}^2 + \tilde{z} = 0$. This does not happen for any value of w.

(ii) (z, ∞), $z \neq \infty$, lies on $z^2 + 2wz + w + z = 0$ if and only if $(z, \tilde{w}) = (z, 0)$ lies on $z^2 + \frac{2z}{\tilde{w}} + \frac{1}{\tilde{w}} + z = 0$, i.e. on $z^2\tilde{w} + 2z + 1 + z\tilde{w} = 0$. This happens when $z = -\frac{1}{2}$, so $(-\frac{1}{2}, \infty)$ lies on the locus.

(iii) (∞, ∞) lies on $z^2 + 2wz + w + z = 0$ if and only if $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on $\frac{1}{\tilde{z}^2} + \frac{2}{\tilde{z}\tilde{w}} + \frac{1}{\tilde{w}} + \frac{1}{\tilde{z}} = 0$, i.e. on $\tilde{w} + 2\tilde{z} + \tilde{z}^2 + \tilde{z}\tilde{w} = 0$. This does happen, so (∞, ∞) lies on the locus.

(b) (i) (∞, w), $w \neq \infty$, lies on $z^2 + 2w^2 + 3 = 0$ if and only if $(\tilde{z}, w) = (0, w)$ lies on $\frac{1}{\tilde{z}^2} + 2w^2 + 3 = 0$, i.e. on $1 + (2w^2 + 3)\tilde{z}^2 = 0$. This does not happen.

(ii) (z, ∞), $z \neq \infty$, lies on $z^2 + 2w^2 + 3 = 0$ if and only if $(z, \tilde{w}) = (z, 0)$ lies on $z^2 + \frac{2}{\tilde{w}^2} + 3 = 0$, i.e. on $(z^2 + 3)\tilde{w}^2 + 2 = 0$. This does not happen.

(iii) (∞, ∞) lies on $z^2 + 2w^2 + 3 = 0$ if and only if $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on $\frac{1}{\tilde{z}^2} + \frac{2}{\tilde{w}^2} + 3 = 0$, i.e. on $\tilde{w}^2 + 2\tilde{z}^2 + 3\tilde{z}^2\tilde{w}^2 = 0$. This does happen, so (∞, ∞) lies on the locus.

(c) (i) (∞, w), $w \neq \infty$, lies on $4wz + z + 1 = 0$ if and only if $(\tilde{z}, w) = (0, w)$ lies on $\frac{4w}{\tilde{z}} + \frac{1}{\tilde{z}} + 1 = 0$, i.e. on $4w + 1 + \tilde{z} = 0$. This happens when $w = -\frac{1}{4}$, so $(\infty, -\frac{1}{4})$ lies on the locus.

(ii) (z, ∞), $z \neq \infty$, lies on $4wz + z + 1 = 0$ if and only if $(z, \tilde{w}) = (z, 0)$ lies on $\frac{4z}{\tilde{w}} + z + 1 = 0$, i.e. on $4z + \tilde{w}(z + 1) = 0$. This happens when $z = 0$, so $(0, \infty)$ lies on the locus.

(iii) (∞, ∞) lies on $4wz + z + 1 = 0$ if and only if $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on $\frac{4}{\tilde{z}\tilde{w}} + \frac{1}{\tilde{z}} + 1 = 0$, i.e. on $4 + \tilde{w} + \tilde{z}\tilde{w} = 0$. This does not happen. ■

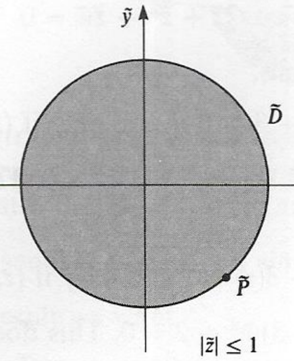
READ B from page 88, line −7 to page 89, line −1.

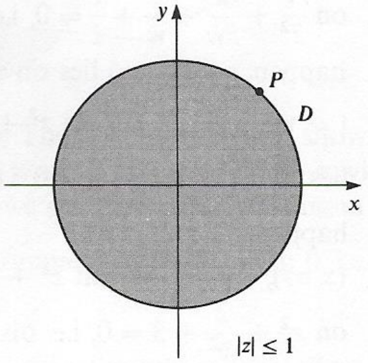
For convenience, we list **B**'s conclusions concerning infinite points:

Q_1 contains (∞, ∞) only;
Q_2 contains $(\infty, -1)$ and $(1, \infty)$, but not (∞, ∞);
Q_3 contains $(\infty, +1)$ and (∞, ∞);
Q_4 contains (∞, ∞) only.

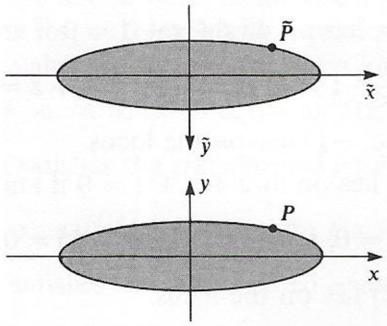
With the inclusion of these infinite points, the loci $f(z, w) = 0$ become compact, and so can be studied topologically using the techniques we have developed earlier in the course. We shall call this procedure of including infinite points the *algebraic way of making the locus compact.*

This algebraic method of dealing with the point ∞ ties up nicely with stereographic projection. Imagine two planes of complex numbers, a z-plane and a $\tilde{z}$-plane. Now pair each non-zero point z with the non-zero point $\tilde{z} = \frac{1}{z}$. The point $z = 0$ is the only point in the z-plane which does not correspond to a point in the $\tilde{z}$-plane. We call it the "point at infinity" in the $\tilde{z}$-plane. We can show that the space obtained by identifying the two planes in this way is a topological sphere. To see this, notice that *every* point in the z-plane is represented *either* by a point with $|z| \le 1$ or by a point with $|z| \ge 1$. In the latter case we have $|\tilde{z}| \le 1$, and so the space obtained by identifying z with $\dfrac{1}{z}$ is the same as the space obtained by taking the unit discs in the z-plane and the $\tilde{z}$-plane and identifying the point $P = z$ on the boundary of D with the point $\tilde{P} = \dfrac{1}{z}$ on the boundary of $\tilde{D}$.

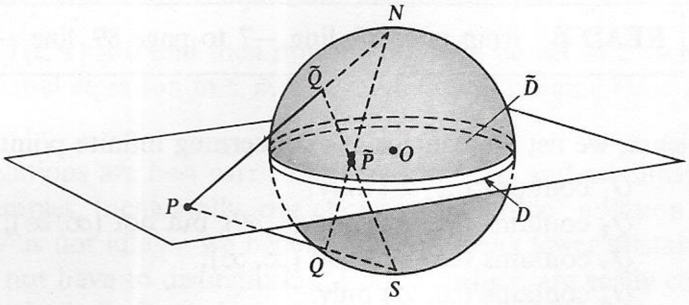




This identification reverses the direction of travel round the boundary, so we can perform the identification physically by turning over the disc $\tilde{D}$ and then glueing its boundary to that of D. The result is clearly a topological sphere.



To make the relationship between this and stereographic projection precise, we can identify the discs $\tilde{D}$ and D with the upper and lower hemispheres of the unit sphere in $\mathbb{R}^3$ as follows.



Map the upper disc, $\tilde{D}$, to the upper hemisphere by projection from S, and map the lower disc, D, to the lower hemisphere by projection from N. Then it can be shown that the stereographic projection of a point P outside the unit disc D in the z-plane is obtained not only by projection from N, but equally by mapping P first to $\tilde{P}$ (in the disc $\tilde{D}$) lying in the $\tilde{z}$-plane and then projecting from S to the point Q in the northern hemisphere. We do not need this result, so we leave the proof as an exercise. (It is fairly clear from the symmetry of the diagram, but if you try it algebraically, remember if P is the point z, so that $\tilde{P}$ is $\frac{1}{z}$, then $\tilde{P}$ actually lies above the complex conjugate of $\frac{1}{z}$ in the disc D, because we turned the disc $\tilde{D}$ over.)

5.2 Loci of Complex Quadratics

This section includes TV and TV-related Audio. The TV programme investigates the loci $w^2 = z$ and $w^2 = z(z - 1)$, and shows that the locus of $w^2 = z$ is a surface (in fact a sphere), whereas the locus of $w^2 = z(z - 1)$ is a pinched sphere. A pinched sphere is a topological sphere in which two points have been identified; for that reason it is not a surface.

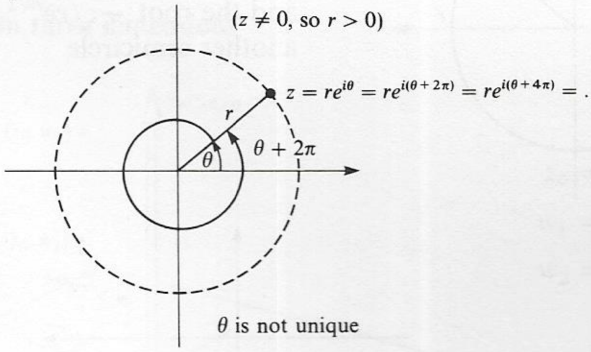
Before watching TV5 you should go through the pre-TV tape section below. This covers some work on complex numbers in preparation for the programme. ALLOW ABOUT 20 MINUTES FOR THIS.

Pre-TV Work

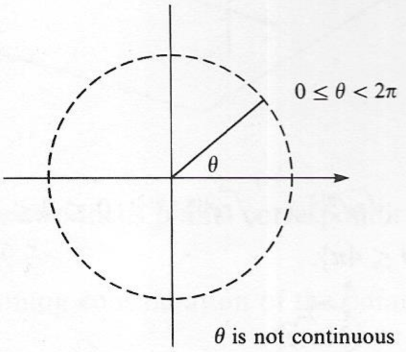
Familiarize yourself with Frame 1, then listen to Track 1, Side 1 of TV-related Audio cassette A2, up to the point where you are told that watching TV5 should be your next activity.

FRAME 1

Complex numbers in terms of polar coordinates


$(z \neq 0$, so $r > 0)$
$z = re^{i\theta} = re^{i(\theta+2\pi)} = re^{i(\theta+4\pi)} = \ldots$
r
θ
$\theta + 2\pi$

θ is not unique



θ is not continuous

START THE TAPE

FRAME 2

The square roots of a non-zero complex number

$w^2 = z \quad \Rightarrow \quad w \pm \sqrt{z}\,?$

Suppose $z = re^{i\theta}$ and $w = se^{i\phi}$;
then $w^2 = s^2e^{2i\theta} = re^{i\phi}$,
which implies $s^2 = r$ and $2\theta = \phi + 2n\pi$,

so $s = \sqrt{r}$ and $\theta = \dfrac{\phi}{2} + n\pi$.

As usual, $\sqrt{r}$ denotes the positive square root of r. Note, however, that for general θ, neither of the square roots of z is a real number, so it is meaningless to talk about them being "positive" and "negative".

The values of w are

$$\sqrt{r}e^{i\theta/2} \quad \text{and} \quad \sqrt{r}e^{i((\theta/2)+\pi)} = -\sqrt{r}e^{i\theta/2}$$
$$(\text{since } e^{i\pi} = -1)$$

So the values of w *are equal and opposite.*

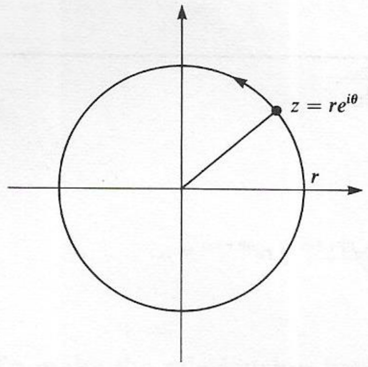
Notation: the symbol $\sqrt{z}$ means either of the square roots of z; that is, either w for which $w^2 = z$.

Warning: since $\sqrt{z}$ is not unique, the assignment $z \longmapsto \sqrt{z}$ is *not* a function.

FRAME 3

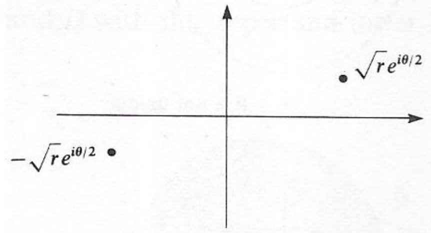
The set of w-values when z varies round a circle

$z = re^{i\theta}$, for fixed $r > 0$, describes a circle as θ goes from 0 to 2π.



The root $\sqrt{r}e^{i\theta/2}$ describes a semicircle, and the root $-\sqrt{r}e^{i\theta/2}$ describes another semicircle.

Draw them:



The w-values are

$$\{\sqrt{r}e^{i\theta/2} : 0 \le \theta \le 2\pi\} \cup \{-\sqrt{r}e^{i\theta/2} = \sqrt{r}e^{i(\pi+\theta/2)} : 0 \le \theta \le 2\pi\}$$
$$= \{\sqrt{r}e^{i\theta/2} : 0 \le \theta \le 4\pi\}.$$

Note

RESTART THE TAPE

FRAME 4

The significant points of Frame 3

When $w^2 = z$, and z lies on the circle centre 0, radius $r > 0$, then:

for each z, there are two values of w.

As z traces out the z-circle once, each w traces out a semicircle. As z traces out the z-circle twice, each w traces out the same w-circle once.

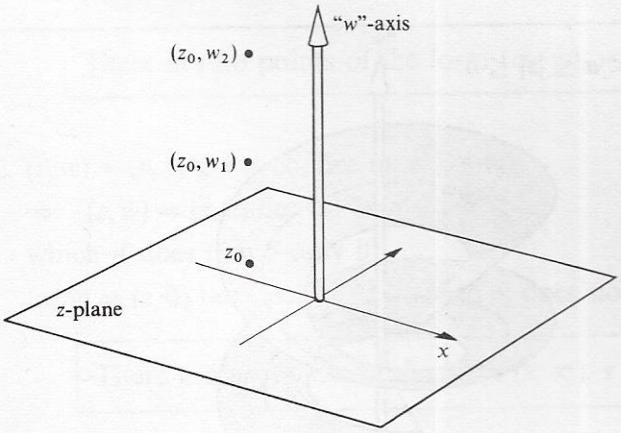
In the television programme, you will see how we can use the above information to build up a useful picture of complex loci, such as $w^2 = z$, in a space of three real dimensions.

TV Overview

After watching TV5, have a quick look again at Frames 1–4 then, before restarting the tape, read Frame 5.

FRAME 5

Complex loci in three dimensions



$$z_0 = re^{i\theta}$$
$$w_1 = \sqrt{r}\,e^{i\theta/2}$$
$$w_2 = -\sqrt{r}\,e^{i\theta/2}$$

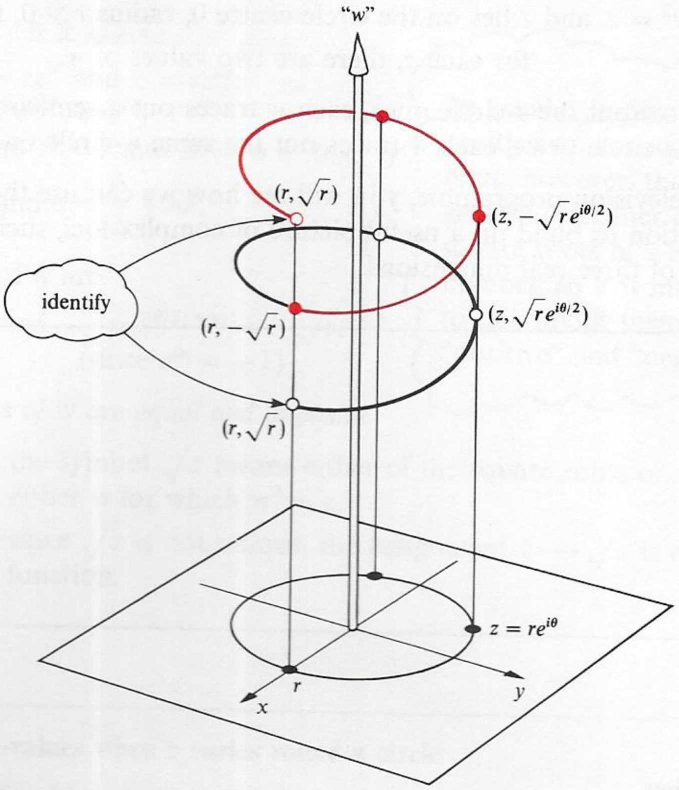
For $w^2 = z$, the two locus points corresponding to $z = z_0 \neq 0$ are plotted vertically above z_0.

(We are postponing consideration of the point $z = 0$.)

RESTART THE TAPE

FRAME 6

The set of points (z, w) for which $z = re^{i\theta}$ (r fixed, $0 \leq \theta \leq 4\pi$)


"w"
$(r, \sqrt{r})$
$(z, -\sqrt{r}e^{i\theta/2})$
identify
$(r, -\sqrt{r})$
$(z, \sqrt{r}e^{i\theta/2})$
$(r, \sqrt{r})$
$z = re^{i\theta}$
r
x
y

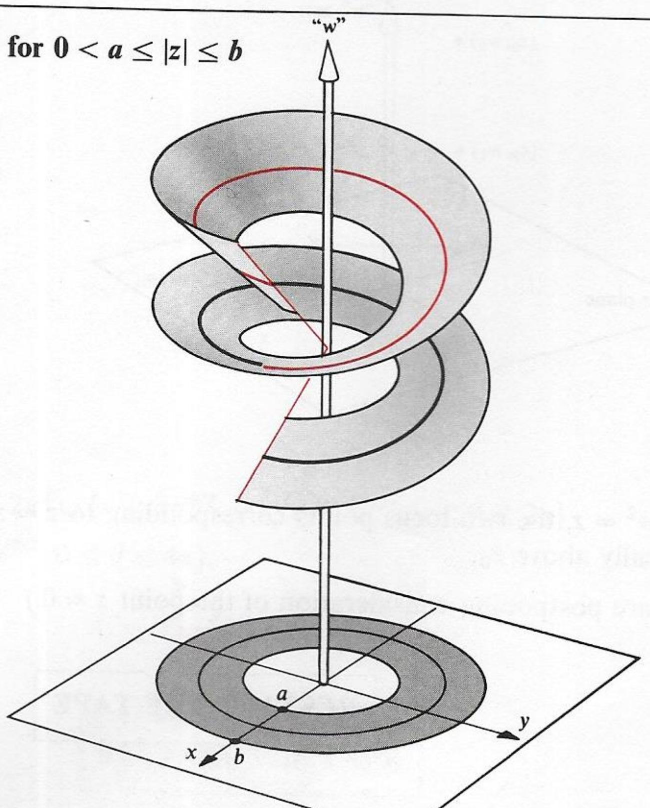
The points (z, w) form a circle on the locus $w^2 = z$.

Check that if either choice of $(z, \sqrt{z})$ goes round the circle upstairs once, z goes round the circle downstairs twice. What happens upstairs if z goes round its circle once?

RESTART THE TAPE

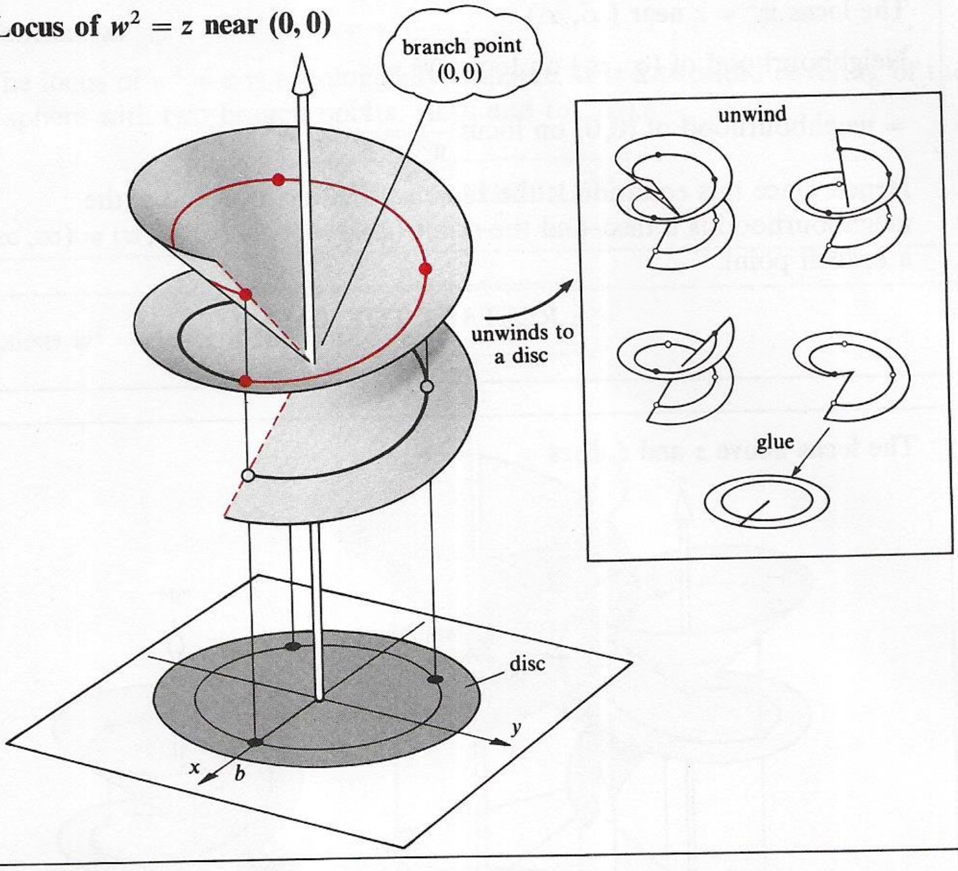
FRAME 7

Locus of $w^2 = z$ for $0 < a \leq |z| \leq b$



FRAME 8

Locus of $w^2 = z$ near $(0, 0)$



FRAME 9

Infinite points on $w^2 = z$

1. $(z, w) = (\infty, w)$, $w \neq \infty$, lies on the locus
$\Leftrightarrow$ $(\tilde{z}, w) = (0, w)$ lies on $w^2 = 1/\tilde{z}$,
which it does if and only if
$(\tilde{z}, w) = (0, w)$ lies on $w^2\tilde{z} = 1$, which it does not.

$\boxed{\text{There are no points of the form } (\infty, w),\ w \text{ finite, on the locus.}}$

2. $(z, w) = (z, \infty)$, $z \neq \infty$, lies on the locus
$\Leftrightarrow$ $(z, \tilde{w}) = (z, 0)$ lies on $1/\tilde{w}^2 = z$,
which it does if and only if
$(z, \tilde{w}) = (z, 0)$ lies on $1 = \tilde{w}^2 z$, which it does not.

$\boxed{\text{There are no points of the form } (z, \infty),\ z \text{ finite, on the locus.}}$

3. $(z, w) = (\infty, \infty)$ lies on the locus
$\Leftrightarrow$ $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on $1/\tilde{w}^2 = 1/\tilde{z}$,
which it does if and only if
$(\tilde{z}, \tilde{w}) = (0, 0)$ lies on $\tilde{z} = \tilde{w}^2$, which it does.

$\boxed{(z, w) = (\infty, \infty) \text{ is a point on the locus.}}$

RESTART THE TAPE

FRAME 10

The locus $w^2 = z$ near (∞, ∞)

Neighbourhood of (∞, ∞) on locus $w^2 = z$

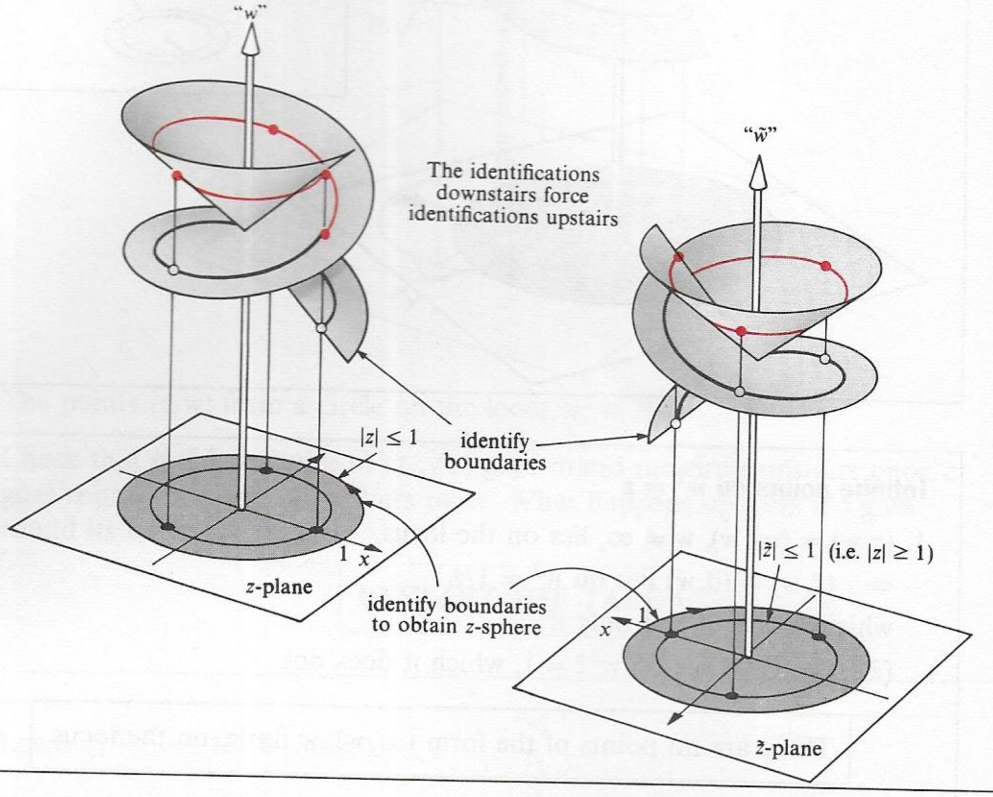
$= \text{neighbourhood of } (0,0) \text{ on locus } \dfrac{1}{\tilde{w}^2} = \dfrac{1}{\tilde{z}}, \text{ i.e. } \tilde{w}^2 = \tilde{z}.$

Hence, since this equation is the same as the one in w and z, the neighbourhood is a disc, and the point $(\tilde{z}, \tilde{w}) = (0,0)$ (i.e. $(z, w) = (\infty, \infty)$) is a branch point.

RESTART THE TAPE

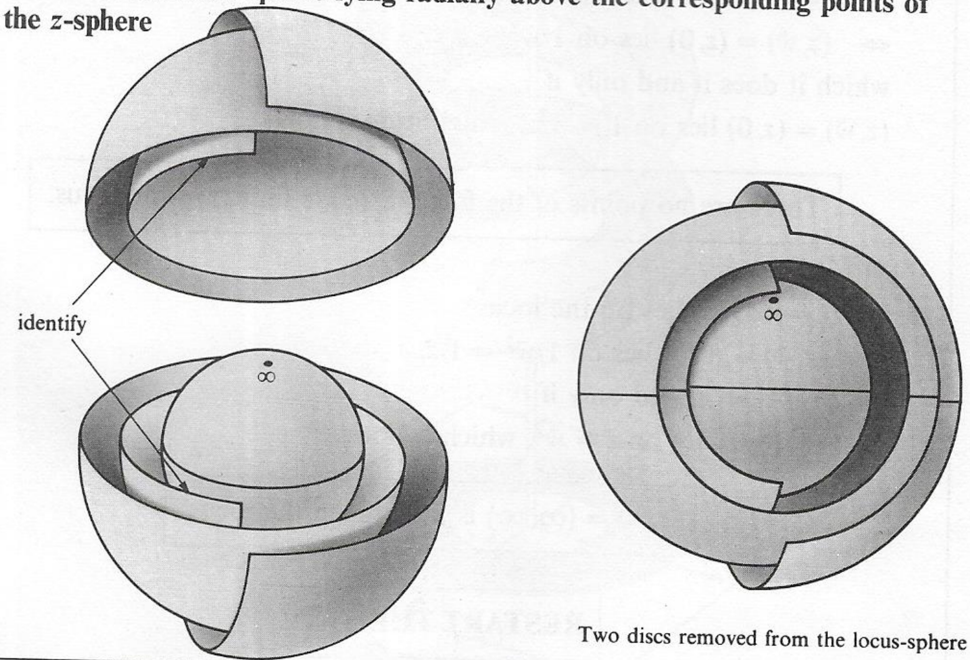
FRAME 11

The locus above z and $\tilde{z}$ discs


"w"
The identifications downstairs force identifications upstairs
"w̃"
|z| ≤ 1
identify boundaries
1
x
z-plane
identify boundaries to obtain z-sphere
|z̃| ≤ 1 (i.e. |z| ≥ 1)
x
1
z̃-plane

FRAME 12

Points of the locus-sphere lying radially above the corresponding points of the z-sphere



Two discs removed from the locus-sphere

FRAME 13

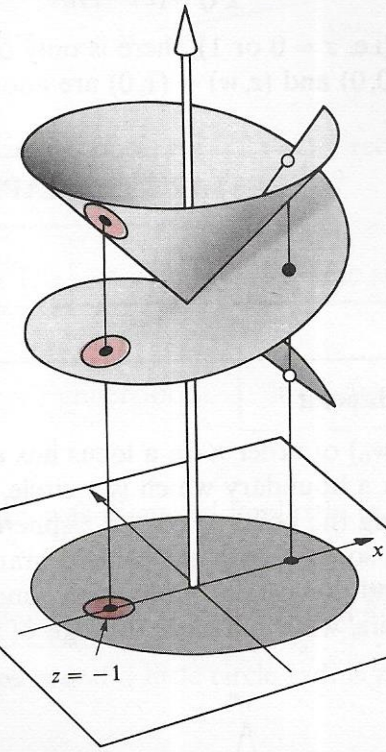
Conclusions

The locus of $w^2 = z$ is topologically a sphere. It is a two-fold covering of the z-sphere with two branch points: $(0,0)$ and (∞, ∞).

RESTART THE TAPE

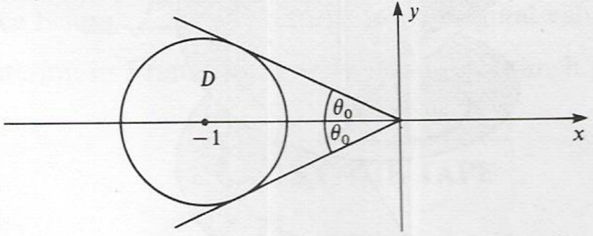
FRAME 14

Locus $w^2 = z$ over a small disc centre $z = -1$



FRAME 15

Polar coordinates in a disc D which does not contain zero (detailed explanation of why there are two discs above the little disc in Frame 14)



For $\pi - \theta_0 \leq \theta \leq \pi + \theta_0$, the label $re^{i\theta}$ for each point z of D is unique. So the assignments

$$re^{i\theta} \longmapsto \sqrt{r}\,e^{i\theta/2}, \quad re^{i\theta} \longmapsto -\sqrt{r}\,e^{i\theta/2}$$

define two continuous functions.

Polar coordinates "work" in D, because D does not contain the origin. They do not "work" on a disc containing the origin, because such a disc contains closed paths which enclose the origin. Around such a path, neither of the assignments above is unique because $\sqrt{z}$ changes continuously

$$\text{from } \sqrt{r}\,e^{i\theta/2} \text{ to } \sqrt{r}\,e^{i(\pi+\theta/2)} = -\sqrt{r}\,e^{i\theta/2}$$

$$(\text{or from } -\sqrt{r}\,e^{i\theta/2} \text{ to } -\sqrt{r}\,e^{i(\pi+\theta/2)} = \sqrt{r}\,e^{i\theta/2}).$$

RESTART THE TAPE

FRAME 16

The locus $w^2 = z(z-1)$

We shall see that this is: a *two-fold covering* of the z-sphere, with *two branch points* and *one pinch point.*

FRAME 17

The locus $w^2 = z(z-1)$

When $z(z-1)$ is not zero, there are two corresponding w-values:

$$\pm\sqrt{(z(z-1))}.$$

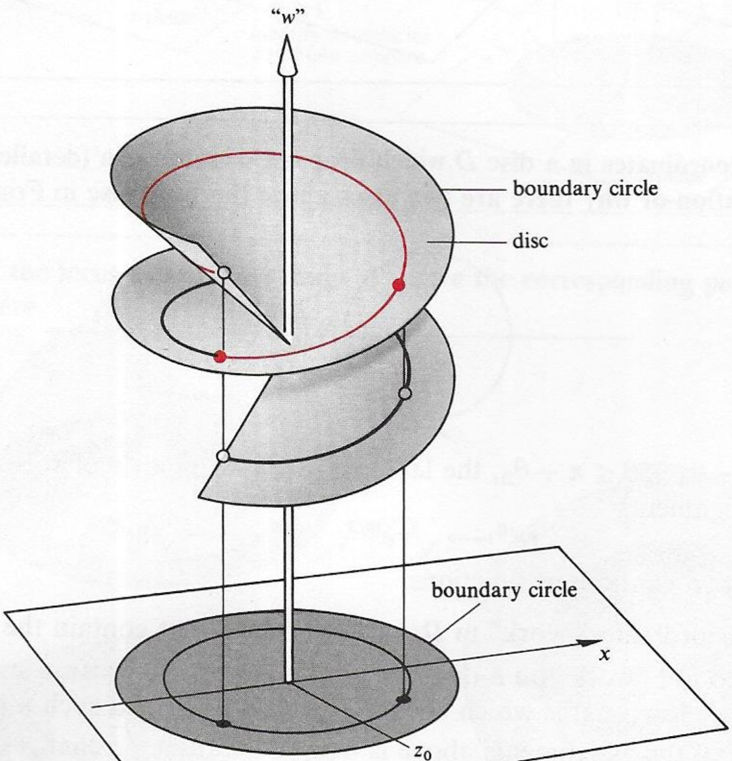
When $z(z-1) = 0$ (i.e. $z = 0$ or 1), there is only one w-value: $w = 0$. So the points $(z, w) = (0, 0)$ and $(z, w) = (1, 0)$ are anomalous.

RESTART THE TAPE

FRAME 18

Criterion for a branch point

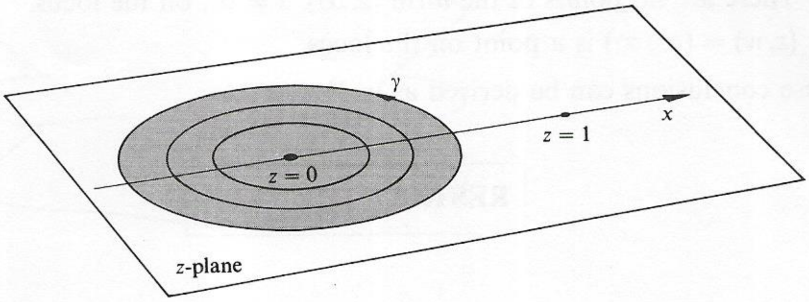
A branch point (z_0, w_0) of order n on a locus has a disc-like neighbourhood, with a boundary which is a circle, that is an n-fold cover of a circle surrounding the point z_0 on the z-sphere. For a conic like $w^2 = z(z-1)$, $n = 2$, so on a path enclosing a branch point, once round upstairs is twice round downstairs. Conversely, once round downstairs is halfway round upstairs, which changes the sign of w.



Before restarting the tape, read Frame 19

FRAME 19

(0, 0) is a branch point of the locus $w^2 = z(z-1)$



As z goes round the circle γ once, $z(z-1) = w^2$ returns to its original value. What about the sign of either value of w?

The sign of $\sqrt{z(z-1)}$ is determined by the sign of $\sqrt{z}$ and $\sqrt{z-1}$.

$\sqrt{z}$ around γ is well understood: it changes sign.

$\sqrt{z-1}$ around γ was discussed implicitly in Frames 14 and 15: its sign does not change. (As z goes round a little circle, radius r, centre zero, $z-1$ goes round a little circle radius r, centre -1... as in Frame 14.)

So the sign of $\sqrt{z(z-1)}$ does change as z goes around γ; z has to go round twice before $\sqrt{z(z-1)}$ returns to its original value.
So, by the criterion in Frame 18, $(z, w) = (0, 0)$ is a branch point.

RESTART THE TAPE

FRAME 20

$(z, w) = (1, 0)$ is also a branch point of the locus $w^2 = z(z-1)$

Write $z = 1 + \zeta$, so $z = 1$ corresponds to $\zeta = 0$ (i.e. we shift the origin to the point $z = 1$). The locus now has equation $w^2 = (1+\zeta)\zeta$, and exactly the same argument as in Frame 19 shows that $\zeta = 0$ is a branch point. (Just replace z by ζ and -1 by $+1$ in Frame 19.)

RESTART THE TAPE

FRAME 21

Infinite points $w^2 = z(z-1)$

(1) There are no points of the form (∞, w), $w \neq \infty$, on the locus.

(2) There are no points of the form (z, ∞), $z \neq \infty$, on the locus.

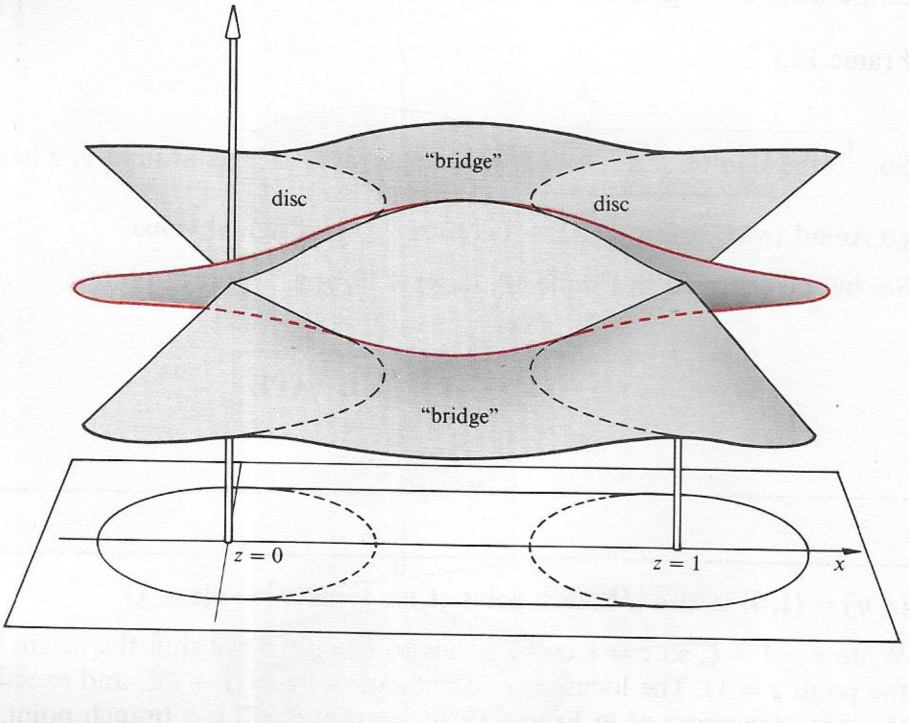
(3) $(z, w) = (\infty, \infty)$ is a point on the locus.

(These conclusions can be derived as in Frame 9.)

RESTART THE TAPE

FRAME 22

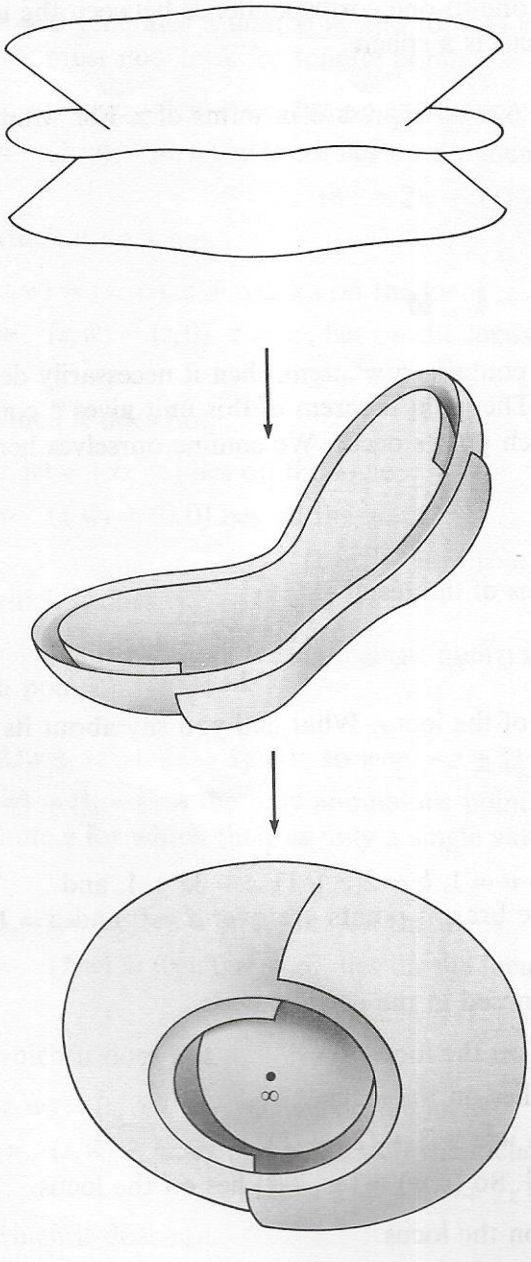
A neighbourhood of (0, 0) and (1, 0) (its boundary also bounds a neighbourhood of (∞, ∞))

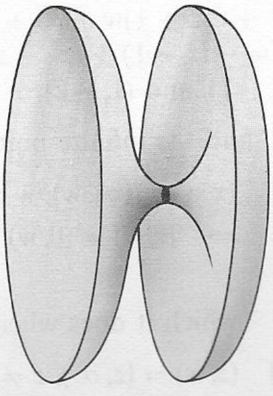


Think about how many components there are in the boundary of this region of the locus.

RESTART THE TAPE

FRAME 23 Deforming the region of the locus in Frame 22 to fit around the *z*-sphere





The missing region over a neighbourhood of (∞, ∞)

What surface would you get if you plugged the hole in the third picture by two separate discs rather than the two pinched discs on the right?

[Hint: Look at Frame 12 to see what surface the third picture gives.]

Post-TV Work

It may happen that a quadratic equation in z and w contains no w^2 term. In this case, there is a continuous one-to-one correspondence between the locus and the z-sphere, and so the locus is a sphere.

SAQ 4 Suppose $2z^2 + 3zw - 4z - w + 6 = 0$. Express w in terms of z. For what value of z is w infinite?

SOLUTION TO SAQ 4

$$w = \frac{2(z^2 - 2z + 3)}{1 - 3z}$$

$$w = \infty \quad \Rightarrow \quad 1 - 3z = 0, \quad \text{i.e. } z = \tfrac{1}{3}. \quad ■$$

If a (non-degenerate) quadratic contains a w^2 term, then it necessarily describes a double cover of the z-sphere. The main theorem of this unit gives a complete analysis of when and where pinch points occur. We confine ourselves here to some examples.

EXAMPLE 2 $2zw + w^2 + 3z + 2w + 1 = 0$.

Express w in terms of z by means of the result

$$aw^2 + bw + c = 0 \quad \Leftrightarrow \quad w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence locate the branch points of the locus. What can you say about its infinite points?

Solution

$w^2 + 2(z + 1)w + 3z + 1 = 0$, so $a = 1$, $b = 2(z + 1)$, $c = 3z + 1$, and $w = -(z + 1) \pm (z^2 - z)^{\frac{1}{2}}$; so the branch points are over $z = 0$ and $z = 1$, i.e. at $(0, -1)$, and $(1, -2)$.

To find the infinite points we proceed in the usual way.

(i) $(z, w) = (\infty, w)$, $w \neq \infty$, lies on the locus

$\Leftrightarrow$ $(\tilde{z}, w) = (0, w)$, $w \neq \infty$, lies on the locus

$$2w + w^2\tilde{z} + 3 + (2w + 1)\tilde{z} = 0,$$

which it does when $w = -\frac{3}{2}$. So $(z, w) = (\infty, -\frac{3}{2})$ lies on the locus.

(ii) $(z, w) = (z, \infty)$, $z \neq \infty$, lies on the locus

$\Leftrightarrow$ $(z, \tilde{w}) = (z, 0)$, $z \neq \infty$, lies on the locus

$$2z\tilde{w} + 1 + 3z\tilde{w}^2 + 2\tilde{w} + \tilde{w}^2 = 0,$$

which it does not.

(iii) $(z, w) = (\infty, \infty)$ lies on the locus

$\Leftrightarrow$ $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on the locus

$$2\tilde{w} + \tilde{z} + 3\tilde{w}^2 + (2\tilde{w} + \tilde{w}^2)\tilde{z} = 0,$$

which it does. So (∞, ∞) lies on the locus.

Thus $z = \infty$ is covered by two points, and so the only anomalous points are $(z, w) = (0, -1)$ and $(z, w) = (1, -2)$. At this stage in your study of this unit, we do *not* expect you to be able to prove that these points are branch points (rather than pinch points).

SAQ 5 Given $w^2 + 3z + 2w + 1 = 0$, express w in terms of z, and find all the values of z for which there is only one corresponding point on the locus. Write down the coordinates of those points.

SAQ 6 Analyse $z^2 + 2zw + w^2 + 2z - 1 = 0$.

SOLUTION TO SAQ 5 $w^2 + 2w + 3z + 1 = 0$, so $w = -1 \pm (-3z)^{\frac{1}{2}}$.

Thus $z = 0$ generates a unique w, and $(0, -1)$ is amongst the locus points we seek. We must now look for infinite points.

(i) $(z, w) = (\infty, w)$, $w \neq \infty$, lies on the locus

$\Leftrightarrow$ $(\tilde{z}, w) = (0, w)$, $w \neq \infty$, lies on the locus

$$(w^2 + 2w + 1)\tilde{z} + 3 = 0,$$

which it does not.

(ii) $(z, w) = (z, \infty)$, $z \neq \infty$, lies on the locus

$\Leftrightarrow$ $(z, \tilde{w}) = (z, 0)$, $z \neq \infty$, lies on the locus

$$1 + 2\tilde{w} + (3z + 1)\tilde{w}^2 = 0,$$

which it does not.

(iii) $(z, w) = (\infty, \infty)$ lies on the locus

$\Leftrightarrow$ $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on the locus

$$(1 + 2\tilde{w} + \tilde{w}^2)\tilde{z} + 3\tilde{w}^2 = 0,$$

which it does.

So (∞, ∞) is the only other anomalous point. (They are, in fact, both branch points.) ■

SOLUTION TO SAQ 6 $w^2 + 2zw + (z^2 + 2z - 1) = 0$, so $w = -z \pm (1 - 2z)^{\frac{1}{2}}$.

So $(z, w) = (\frac{1}{2}, -\frac{1}{2})$ is the only anomalous point with z finite — i.e. $z = \frac{1}{2}$ is the only finite z for which there is only a single value of w. Now we look for infinite points.

(i) $(z, w) = (\infty, w)$, $w \neq \infty$, lies on the locus

$\Leftrightarrow$ $(\tilde{z}, w) = (0, w)$, $w \neq \infty$, lies on the locus

$$w^2\tilde{z}^2 + 2\tilde{z}w + 1 + 2\tilde{z} - \tilde{z}^2 = 0,$$

which it does not.

(ii) $(z, w) = (z, \infty)$, $z \neq \infty$, lies on the locus

$\Leftrightarrow$ $(z, \tilde{w}) = (z, 0)$, $z \neq \infty$, lies on the locus

$$1 + 2z\tilde{w} + (z^2 + 2z - 1)\tilde{w}^2 = 0,$$

which it does not.

(iii) $(z, w) = (\infty, \infty)$ lies on the locus

$\Leftrightarrow$ $(\tilde{z}, \tilde{w}) = (0, 0)$ lies on the locus

$$\tilde{z}^2 + 2\tilde{z}\tilde{w} + (1 + 2\tilde{z} - \tilde{z}^2)\tilde{w}^2 = 0,$$

which it does.

So (∞, ∞) is another anomalous point. Once again, $(z, w) = (\frac{1}{2}, -\frac{1}{2})$ and (∞, ∞) are, in fact, branch points, as we shall see in the next section. ■

5.3 Introduction to Complex Conics

A real conic section is a locus defined by a second degree equation

$$\{(x, y) : ax^2 + bxy + cy^2 + dx + ey + f = 0\}$$

where a, b, c, d, e, f are real numbers, and not all a, b, c are zero.

You may recall that real conic sections can be classified as ellipses, parabolas, or hyperbolas, or as certain so-called "degenerate" conics such as points or pairs of lines. This is described in *M101 Block IV*, and treated more fully in *M203 Units 3, 17* and *21*. It is also summarized in **B** pages 85–6.

In this section we are concerned with the *complex* case. We shall try to represent such loci as surfaces, taking z to lie on the Riemann sphere.

DEFINITION A *complex conic* is a locus

$$\{(z, w): Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0\},$$

where A, B, C, D, E, F are complex numbers, not all of A, B, C are zero, and z and w are allowed to take complex values.

The equation

$$Q(z, w) = Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0$$

may be written as

$$Cw^2 + (Bz + E)w + Az^2 + Dz + F = 0,$$

and, considered as a quadratic in w for each value of z it always has two solutions, provided $C \neq 0$. These solutions will, in general, be distinct complex numbers; for $z = z_0$, write w_0^+ and w_0^- for the solutions of

$$Cw^2 + (Bz_0 + E)w + Az_0^2 + Dz_0 + F = 0.$$

Using the usual formula for the roots of a quadratic, we have

$$w_0^+ = \frac{1}{2C}\{-(Bz_0 + E) + [(Bz_0 + E)^2 - 4C(Az_0^2 + Dz_0 + F)]^{\frac{1}{2}}\},$$

$$w_0^- = \frac{1}{2C}\{-(Bz_0 + E) - [(Bz_0 + E)^2 - 4C(Az_0^2 + Dz_0 + F)]^{\frac{1}{2}}\}.$$

So we can cover the Riemann sphere Σ by a two-sheeted surface, $\bar{\Sigma}$, consisting of the pairs of distinct points w_0^+ and w_0^-, *except* possibly for the points z_0 where

$$(Bz_0 + E)^2 = 4C(Az_0^2 + Dz_0 + F). \qquad (*)$$

Of course, it remains to examine the points of $\bar{\Sigma}$ which correspond to ∞ in Σ, and to points z_0 which satisfy $(*)$.

The point of crucial importance is to understand how the two sheets of $\bar{\Sigma}$ fit together to form either a single surface or a surface with a pinch point. This is first discussed in **B** in the simplest case, $Q(z, w) = z^2 - w = 0$. The answer hinges on the ambiguous notion of the square root of a complex number. In the next reading passage, **B** first reviews real conic sections, specifies what it is for a conic to be nondegenerate, and then briefly reviews some properties of complex numbers, before discussing this example (at the foot of page 87).

READ B from page 85 to page 88, line -7.

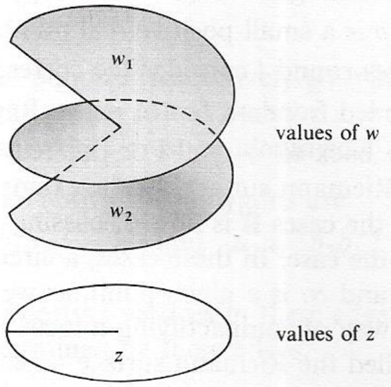
NOTES (i) *Page 86, line* -7

In the definition of degenerate, conditions 1 and 2 are in fact special cases of condition 3, as follows:

(1) implies the equation is $Dz + Ew + F = 0$, which can be written $(Dz + Ew + F)\cdot 1 = 0$, exhibiting it as linear times constant;

(2) implies (taking $A = B = D = 0$) that the equation is $w^2 + Ew + F = 0$, which can be factored since it is just a quadratic in w.

(ii) *Page 88, line 18*
Consider what happens as z crosses the negative real axis at the point -1. At points near -1 and above the real axis, θ is near π, so w_1 is near i (and w_2 near $-i$). However, at points near -1 *below* the real axis, θ is near $-\pi$ and so w_1 is near $-i$ (and w_2 near i). Thus as z moves on a path through -1, the path traced by w_1 jumps from i to a point near $-i$ (with argument near $-\pi/2$). To ensure that w_1 changes continuously, we have to replace w_1 by w_2 (which is near i for z near -1 and in the third quadrant) as z crosses the axis. Moreover, as z crosses the axis, the value w_2 must be replaced by w_1. The sheets are joined together as in the diagram below. This can be done without self-intersection only in $\mathbb{C}^2$ (which, remember, has four real dimensions).



▲

It remains for **B** to study what happens on the locus $w^2 = z$ when $z = \infty$. We have already considered the passage in which this example occurs as $Q_1(z, w) = w^2 - z$, and we found that the only infinite point on the locus is (∞, ∞).

At this point, **B** has shown: that to every finite non-zero z, there correspond two distinct points on the complex locus $w^2 = z$; that if $z = 0$, then $w = 0$; and that if $z = \infty$, then $w = \infty$. He has shown that the locus fits together to form a surface. He has not shown what surface the locus is topologically; this will be done later (page 94). Next, he deals with the general quadratic equation in z and w. The only new fact that arises is that infinite points can be pinch points.

READ B page 89, line -1 to page 91, line 12.

NOTES (i) *Page 90, lines 16–20*
$(-E/B, \infty)$ lies on the locus if $Bz + E = -\tilde{w}(Az^2 + Dz + F)$ is satisfied by $(-E/B, 0)$, which it is.

Suppose $A \neq 0$; then (∞, w) lies on the locus if $(0, w)$ lies on $w(B + E\tilde{z})\tilde{z} = -(A + D\tilde{z} + F\tilde{z}^2)$, which is false for finite w. (∞, ∞) does lie on the locus, since $(0, 0)$ satisfies $(B + E\tilde{z})\tilde{z} = -(A + D\tilde{z} + F\tilde{z}^2)\tilde{w}$.

Suppose $A = B = 0$; then (∞, w) lies on the locus $Ew = -(Dz + F)$ if $(0, w)$ lies on $Ew\tilde{z} = -(D + F\tilde{z})$, which is false for finite w (since the conic is nondegenerate). The point (∞, ∞) lies on the locus if $E\tilde{z} = -(D + F\tilde{z})\tilde{w}$ is satisfied by $(0, 0)$, which it is.

Suppose $A = 0$ but $B \neq 0$. Then (∞, w), $w \neq \infty$, lies on the locus if $(0, w)$ lies on $w(B + E\tilde{z}) = -(D + F\tilde{z})$, i.e. if $w = -D/B$. However, $(B + E\tilde{z}) = -(D + F\tilde{z})\tilde{w}$ is not satisfied by $(0, 0)$, so (∞, ∞) does not lie on the locus.

(ii) *Page 90, line -5*
The complex parabola is discussed on page 94.

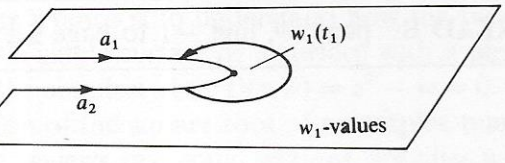
(iii) *Page 90, line −1*
B's formula is trivially wrong, and should read:

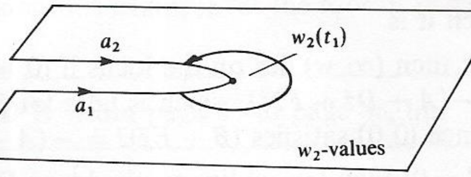
$$Q(z,w) = C\left(w + \frac{Bz+E}{2C} - \frac{Kz+H}{2C}\right)\cdot\left(w + \frac{Bz+E}{2C} + \frac{Kz+H}{2C}\right). \quad ▲$$

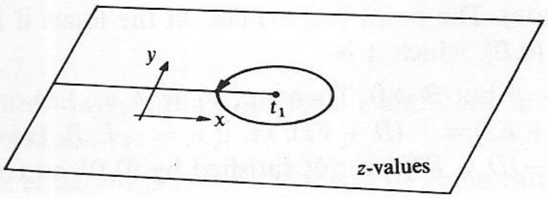
Blackett has now shown (on the assumption that $B^2 - 4AC \neq 0$) that every point other than ∞, t_1 and t_2 on the Riemann sphere is covered by two points of Σ. He has yet to show that these pairs of points define a connected surface, and his discussion occupies the next two and a half pages. In dealing with $w^2 = z$, we saw that the two determinations of w ($+\sqrt{z}$ and $-\sqrt{z}$) were interchanged on crossing the negative real axis. There is another way to see this. Consider a loop l, encircling the origin in the z-plane, which starts and finishes at $z = \rho$, say, where ρ is a small positive real number. Pick a value of $\sqrt{\rho}$, say $+\sqrt{\rho}$, and as z moves round l consider the corresponding value of w starting from $+\sqrt{\rho}$. You are led from $+\sqrt{\rho}$ to $-\sqrt{\rho}$. Repeating l leads, in the w-plane, from $-\sqrt{\rho}$ back to $+\sqrt{\rho}$. For this reason, we have called $w = 0$ a branch point of the Riemann surface. In the example $w^2 = z$, $w = \infty$ is also a branch point, but in the cases **B** is now discussing ($B^2 - 4AC \neq 0$, $A \neq 0$, $C \neq 0$), this is never the case. In these cases, a circuit around ∞ does not exchange the leaves, and ∞ is a pinch point, as we shall see. For this reason, **B** introduces a second way of compactifying a locus in z and w which will *always* yield a surface — called the Riemann surface — corresponding to the locus.

READ B from page 91, line 13 to the end of Section 3.1.

NOTES (i) *Page 92, lines 1–7*
If $z - t_1 = \rho e^{i\theta}$ for a small fixed ρ ($\rho < \frac{1}{2}|t_1 - t_2|$), then $\sqrt{z - t_2}$ never changes sign, and so, near $z = t_1$, w_1 behaves like $\sqrt{z - t_1}$, as far as the ambiguous sign is concerned. Similarly, w_2 behaves like $-\sqrt{z - t_1}$ near $z = t_1$. This means that a circuit of z around t_1 exchanges the sheets of w_1 and w_2 along their edges (the points corresponding to z lying on a). Schematically we may represent this as:

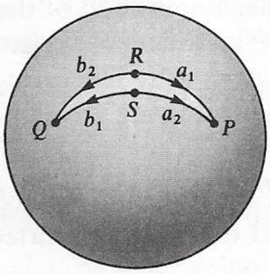


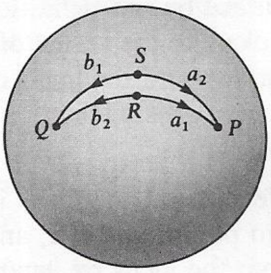




$$w_1(t_1) = w_2(t_1) = -\frac{(Bt_1 + E)}{2C}$$

The picture of each sheet over b is the same, so the sheets (thought of as covering the z-sphere) look like:





The cuts extend from the points P(respectively Q) lying over $z = t_1$ (respectively $z = t_2$) to the points R and S above $z = \infty$. The picture makes clear the way in which the edges must be identified.

The edge equations of the above two polygons are

$$a_1 a_2^{-1} b_1 b_2^{-1} = 1 \quad \text{and} \quad a_2 a_1^{-1} b_2 b_1^{-1} = 1.$$

EXAMPLE 3 You might like to check, by inserting vertices in these edge equations, that precisely four vertices are determined by these edges.

Solution

$(a_1 a_2^{-1}) = (a_2 a_1^{-1}) \qquad (=P)$

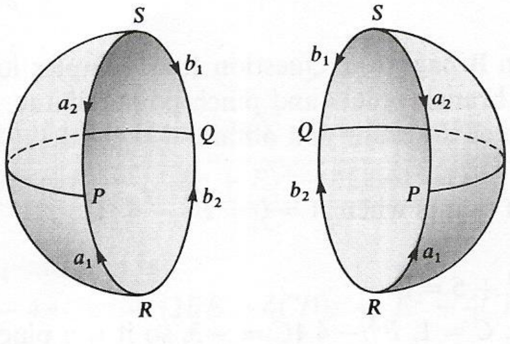
$(a_2^{-1} b_1) = (b_1^{-1} a_2) \qquad (=S)$

$(b_1 b_2^{-1}) = (b_2 b_1^{-1}) \qquad (=Q)$

$(b_2^{-1} a_1) = (a_1^{-1} b_2) \qquad (=R)$ □

This is the crucial step in the new method: the locus is analysed to produce a system of edge equations. The locus is cut up according to the edge equations, and classified by the techniques of *Unit 4*. The point to remember is that this method *always* yields a surface, because extra points are added to make this true. For this reason, the Riemann surface is not necessarily the same as the locus which we studied by means of the theory of infinite points. In this example, R and S are not identified, since there is no geometric reason to identify them.

Opening up the cuts, we can see what these sheets look like topologically:



$$P = (t_1, w_1(t_1)) = (t_1, w_2(t_1))$$
$$Q = (t_2, w_1(t_2)) = (t_2, w_2(t_2))$$

We can also see that these two hemispheres can be joined together to give a topological sphere.

NOTE (ii) *Page 93, lines −9 to −4*

Blackett should have stressed this point more forcefully: the Riemann surface obtained from the edge equations *is* a surface, but the *locus* may have pinch points. When the locus has pinch points, it is not strictly a surface by our definition, because the neighbourhood of the pinch point looks like the vertex of a double cone. The Riemann surface is a genuine surface, and the locus can be obtained from it by identifying a pair of points. This distinction will be worth remembering throughout the next two units.

We say that the locus is obtained from the Riemann surface by *pinching* two points together, and conversely that the Riemann surface is obtained from the locus by *unpinching* the pinch point.

For your convenience, we restate **B**'s main Theorem on pages 93–4:

A nondegenerate complex conic $Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0$ is a sphere if $A = 0$ or $C = 0$ or $B^2 - 4AC = 0$, and it is a pinched sphere otherwise. The branch points and pinch points occur as follows:

1. $A = 0$, $C \neq 0$. The locus and the Riemann sphere coincide; each contains two points (z, w) for every z except two values of z which locate the branch points. (∞, ∞) is a branch point if and only if $B^2 - 4AC = 0$, i.e. if and only if $B = 0$.
2. $C = 0$, any A. The locus and the Riemann surface coincide and each contains a *unique* point (z, w) for each z. There are no branch points.
3. $A \neq 0$, $C \neq 0$ but $B^2 - 4AC = 0$. The locus and the Riemann surface coincide; each contains two points for every z except for the two branch points, one of which is (∞, ∞).
4. In all other cases, for finite values of z there are two points on the locus and on the Riemann surface except at the two branch points. The point (∞, ∞) on the locus is invariably a pinch point. The Riemann surface contains two points which correspond to the point $z = \infty$ on the Riemann sphere.

Observe that in cases 1, 3 and 4, the Riemann surface corresponding to the locus is a double cover of the z-sphere branched over two points, whereas in case 2 it is a single cover (which is necessarily unbranched). Moreover, we have now proved that the only anomalous points on a complex quadratic locus are branch points or pinch points, as we claimed in Section 5.1. You might like to compare this with your answers to the earlier SAQs.

Unless you are explicitly asked to do so in a question, it will not be necessary to analyse a given locus in such detail again, and it will be enough to quote this theorem. ▲

SAQ 7 Analyse the loci in **B** page 106, Question 1, as complex loci (write z for x and w for y). Locate the branch points and pinch points, if any, and state the topological nature of the locus you obtain.

SAQ 8 Locate the branch points when $A = 0 = B^2 - 4AC$.

SOLUTION TO SAQ 7

(a) $z^2 + zw + w^2 + 5 = 0$

$A = 1, B = 1, C = 1, B^2 - 4AC = -3$, so it is a pinched sphere.

As an equation in w, $w^2 + zw + (z^2 + 5) = 0$, it has repeated roots when $z^2 = 4(z^2 + 5)$, i.e. at $z = \pm\sqrt{-20/3}$. These are branch points.

When $z = \infty$, consider $\tilde{z} = 1/z$.

$1 + \tilde{z}w + (w^2 + 5)\tilde{z}^2 = 0$ has no solution in finite w when $\tilde{z} = 0$.

When $w = \infty$, consider $\tilde{w} = 1/w$.
$z^2\tilde{w}^2 + z\tilde{w} + 1 + 5\tilde{w}^2 = 0$ has no solution in finite z when $\tilde{w} = 0$.
For $w = z = \infty$,
$\tilde{w}^2 + \tilde{z}\tilde{w} + (1 + 5\tilde{w}^2)\tilde{z}^2 = 0$ has the solution $\tilde{w} = 0$ when $\tilde{z} = 0$, so the point (∞, ∞) lies on the surface, and it is a pinch point.

(b) $z^2 + zw + w^2 - 5 = 0$ is another pinched sphere, with branch points at $z = \pm\sqrt{20/3}$ and a pinch point at (∞, ∞).

(c) $z^2 + zw + w^2 = 0$ is not a pinched sphere, because it is degenerate. In terms of **B**'s analysis, page 90, $d(z) = -3z^2$, so

$$z^2 + zw + w^2 = (w + \tfrac{1}{2}(1 - \sqrt{-3})z)(w + \tfrac{1}{2}(1 + \sqrt{-3})z).$$

(d) $z^2 + 2zw + w^2 + z = 0$ is a sphere, since $B^2 - 4AC = 0$. The branch points are (∞, ∞) and $(0, 0)$.

(e) $z^2 + 3zw + w^2 = 0$ is not a pinched sphere, because it is degenerate. Indeed,

$$z^2 + 3zw + w^2 = \left(w + \frac{3 - \sqrt{5}}{2}z\right)\left(w + \frac{3 + \sqrt{5}}{2}z\right). \quad ■$$

SOLUTION TO SAQ 8 We must have $B = 0$, $C \neq 0$, $D \neq 0$ since the conic is nondegenerate, so its equation is

$$Cw^2 + Dz + Ew + F = 0.$$

Thus $w = \dfrac{-E \pm (E^2 - 4C(Dz + F))^{1/2}}{2C}$, so one branch point is at

$$(z, w) = \left(\frac{E^2 - 4CF}{4CD}, \frac{-E}{2C}\right).$$

The other must be over $z = \infty$, by the Theorem. If (∞, w), $w \neq \infty$, lies on the conic, then

$$(Cw^2 + Ew + F)\tilde{z} + D = 0.$$

This has no solution for finite w when $\tilde{z} = 0$, so the other branch point must be (∞, ∞). ■

5.4 Further Self-assessment Questions

SAQ 9 **B** page 106, Question 2.
SAQ 10 **B** page 106, Question 3.
SAQ 11 **B** page 106, Question 4.
SAQ 12 **B** page 106, Question 5.

SOLUTION TO SAQ 9 From the ordinary formula (on **B** page 90), the solution for w in terms of z is

$$w = \frac{-(Bz + E) \pm \sqrt{(Bz + E)^2 - 4C(Az^2 + Dz + F)}}{2C}.$$

The term under the square root is

$$(B^2 - 4AC)z^2 + (2BE - 4CD)z + (E^2 - 4CF),$$

and this is a perfect square if

$$(BE - 2CD)^2 = (B^2 - 4AC)(E^2 - 4CF).$$

Since $C \neq 0$, this reduces to

$$CD^2 + B^2F + AE^2 - BDE - 4ACF = 0.$$

The converse is also true. ■

SOLUTION TO SAQ 10 Points of the form (∞, w) on $w^2 - wz + z + 1 = 0$ are solutions to $w^2\tilde{z} - w + 1 + \tilde{z} = 0$ when $\tilde{z} = 0$, i.e. they are solutions to $w = 1$. So $(\infty, 1)$ lies on the conic. (∞, ∞) lies on the conic if and only if $(0, 0)$ satisfies $\tilde{z} - \tilde{w} + \tilde{w}^2 + \tilde{z}\tilde{w}^2 = 0$, which it does. So (∞, ∞) also lies on the conic. If z determines a unique point (z, w) on the conic, then the quadratic $w^2 - zw + (z + 1) = 0$ in w has a repeated root. This occurs when

$$z^2 - 4(z+1) = 0, \quad \text{i.e. when } z = 2 \pm 2\sqrt{2}.$$

The point $z = \infty$ must be treated separately, but we have seen that there are two corresponding values of w: 1 and ∞. ■

SOLUTION TO SAQ 11

(a) (∞, w) is on the locus for $w = \pm 2i$.

(z, ∞) is on the locus for $z = -1 \pm 2i$.

(∞, ∞) is not on the locus.

(b) (∞, w) is on the locus for $w = -2$.

(z, ∞) is on the locus for $z = \pm\sqrt{3}i$.

(∞, ∞) is on the locus. ■

SOLUTION TO SAQ 12 For a fixed finite value of z, the equation

$$(z^2 + 1)w^2 + z^2 w + \tfrac{1}{2}z^2 - \tfrac{1}{4} = 0$$

has two roots unless it is a perfect square, i.e. unless

$$z^4 = (z^2 + 1)(2z^2 - 1).$$

Simplifying gives

$$z^4 + z^2 - 1 = 0;$$

so $z^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, which gives four values of z (two of which are real).

(z, ∞) is on the locus for $z = \pm i$.

(∞, w) is on the locus for $w = \frac{1}{2}(-1 \pm i)$;

so ∞ is covered by two points. ■

5.5 Continuity (Part 2) (Optional)

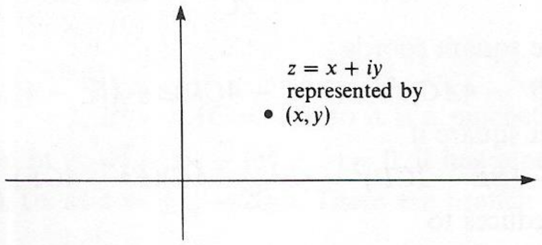
This section takes the idea of continuity discussed in Section 3.4, and generalizes it to spaces other than subsets of $\mathbb{R}^n$ (such as the space of complex numbers, $\mathbb{C}$, the Riemann sphere, $\mathbb{C} \cup \{\infty\}$, and the "ray" model of $\mathbb{RP}^2$). Finally, the three models of $\mathbb{RP}^2$ are shown to be homeomorphic.

The commentary for this section is the second part of Audio Cassette B.

START THE TAPE

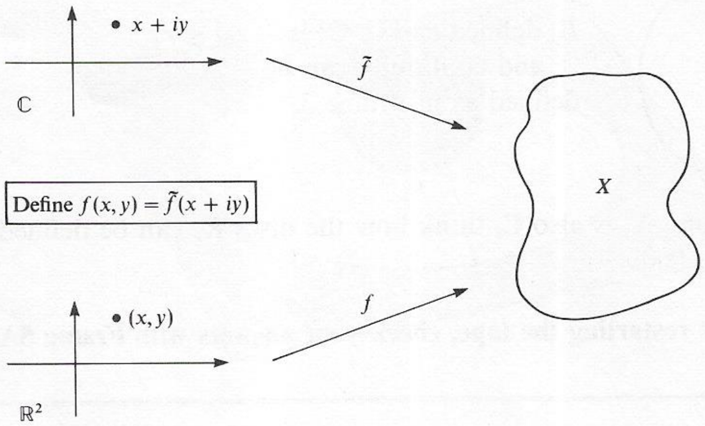
FRAME 1

The complex numbers ℂ



FRAME 2

$f:\mathbb{R}^2 \longrightarrow X$ related to $\tilde{f}:\mathbb{C}\longrightarrow X$



$x+iy$
$\mathbb{C}$
$\tilde{f}$
X
(x,y)
f
$\mathbb{R}^2$

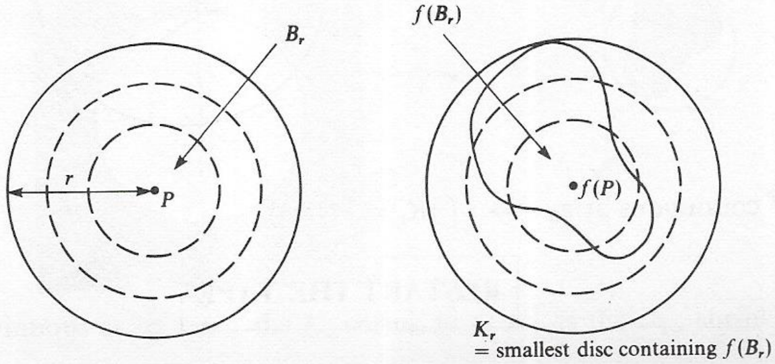
$\boxed{\text{Define } f(x,y)=\tilde{f}(x+iy)}$

Define $\tilde{f}$ continuous $\Leftrightarrow$ f continuous.

FRAME 3

Reminder of definitions of continuity for $f:\mathbb{R}^2\longrightarrow X$

$\bigcap B_r = \{P\}$

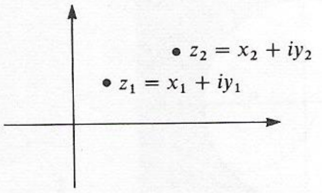


B_r
r
P
$f(B_r)$
$f(P)$
K_r
= smallest disc containing $f(B_r)$

f continuous at P $\Leftrightarrow$ $\bigcap K_r = \{f(P)\}$.

FRAME 4

The "distance" between two complex numbers



$z_2 = x_2 + iy_2$
$z_1 = x_1 + iy_1$

The distance between z_1 and z_2 is defined as $|z_1 - z_2|$.

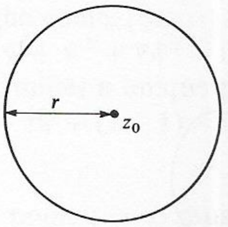
But $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

So $|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ = distance in $\mathbb{R}^2$ between (x_1, y_1) and (x_2, y_2).

RESTART THE TAPE

FRAME 5

Continuity of $\tilde{f}: \mathbb{C} \longrightarrow X$ at z_0



B_r defined as $\{z \in \mathbb{C}: |z - z_0| \leq r\}$
… and continuity can be defined as in Frame 3.

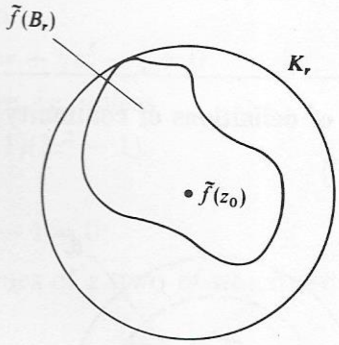
If the codomain, X, is also $\mathbb{C}$, think how the discs K_r can be defined in terms of $|z - \tilde{f}(z_0)|$.

Before restarting the tape, check your answers with Frame 5A.

FRAME 5A

Solution for Frame 5 (definition of K_r for codomain $\mathbb{C}$)

Define K_r as
$\{z \in \mathbb{C}: |z - \tilde{f}(z_0)| \leq S_r\}$
where S_r is the smallest number such that $\tilde{f}(B_r) \subset K_r$.



Then $\tilde{f}$ continuous at z_0 $\Leftrightarrow$ $\bigcap K_r = \{\tilde{f}(z_0)\}$.

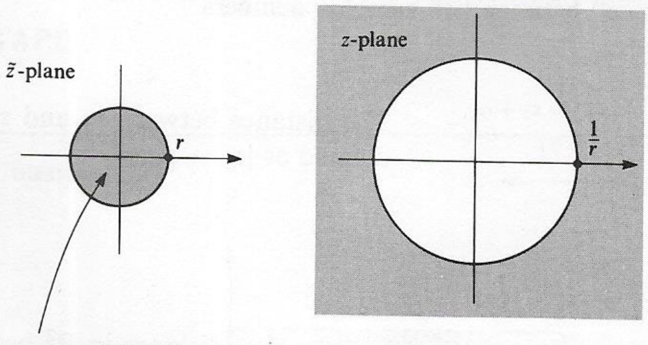
RESTART THE TAPE

FRAME 6

Continuity at ∞ of $f: \mathbb{C} \cup \{\infty\} \longrightarrow X$

(a) Discs at ∞ Put $\tilde{z} = \dfrac{1}{z}$.

$$\tilde{z} = 0 \longleftrightarrow z = \infty$$



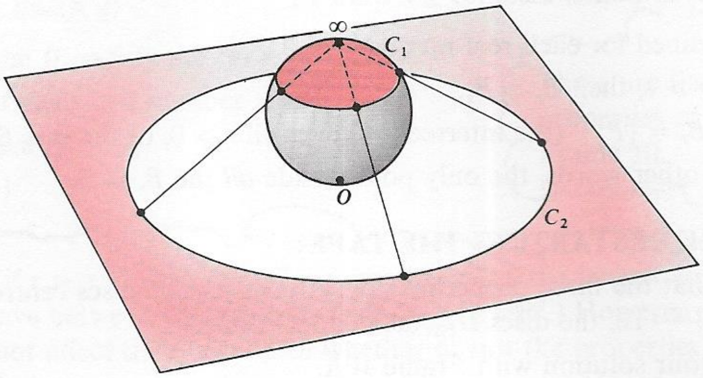
$\tilde{B}_r$ = small disc round $\tilde{z} = 0$ $\leftrightarrow$ outside of large disc plus ∞

$$D_{1/r} = \left\{z \in \mathbb{C}: |z| \geq \frac{1}{r}\right\} \cup \{\infty\}.$$

RESTART THE TAPE

FRAME 7

Disc centre ∞ on the Riemann sphere

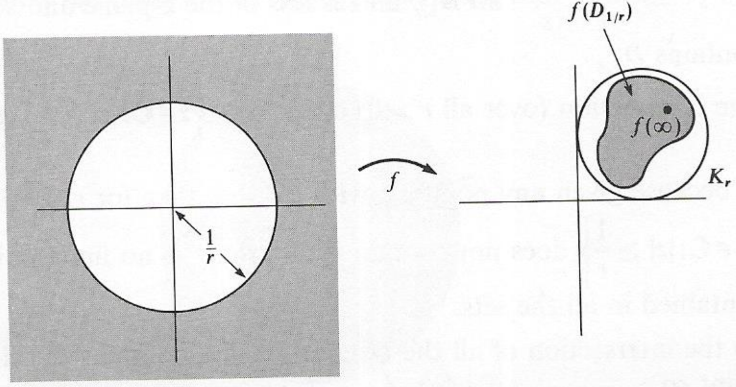


FRAME 8

Continuity at ∞ of $f\colon \mathbb{C} \cup \{\infty\} \longrightarrow X$

(b) The sets $D_{1/r}$ and the discs K_r

$$D_{1/r} = \left\{z \in \mathbb{C} : |z| \geq \frac{1}{r}\right\} \cup \{\infty\}$$

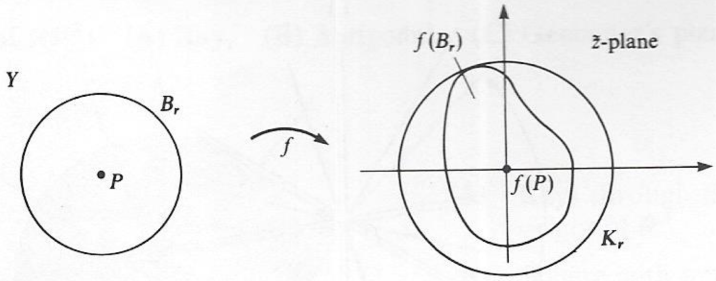


f continuous at ∞ $\Leftrightarrow$ the K_r shrink to $f\{\infty\}$ as the $D_{1/r}$ shrink to ∞.

FRAME 9

Continuity at P of $f\colon Y \longrightarrow \mathbb{C} \cup \{\infty\}$ when $f(P) = \infty$

(c) The discs K_r in this case



$K_r = \{\tilde{z} \in \mathbb{C} : |\tilde{z}| \leq S_r\}$, where S_r is the smallest number such that $f(B_r) \subset K_r$.

See if you can re-write this in terms of z $\left(\text{instead of } \tilde{z} = \frac{1}{z}\right)$.

RESTART THE TAPE

FRAME 10

Properties of the discs B_r in the definition of continuity

(B_r = set of points distance $\le r$ from P)

(1) Defined for each real number $r > 0$.

(2) If $r > s$, then $B_r \supset B_s$.

(3) $\bigcap B_r = \{P\}$ (the intersection, over all $r > 0$, of the sets $B_r = \{P\}$). In other words, the only point inside *all* the B_r is P.

BEFORE RESTARTING THE TAPE:

Check that the three properties are satisfied for the discs centred on ∞ in $\mathbb{C} \cup \{\infty\}$ — i.e. the discs $D_{1/r}$ defined in Frame 6.

Check your solution with Frame 10A.

FRAME 10A

Solution for Frame 10

$$D_{1/r} = \left\{z \in \mathbb{C} : |z| \ge \frac{1}{r}\right\} \cup \{\infty\}$$

(1) Trivial for $D_{1/r}$.

(2) $r > s \;\Rightarrow\; \frac{1}{r} < \frac{1}{s}$. So $D_{1/r}$ misses less of the z-plane and so contains $D_{1/s}$.

(3) The intersection (over all $r > 0$) of the sets $\left\{z \in \mathbb{C} : |z| \ge \frac{1}{r}\right\}$ is empty...

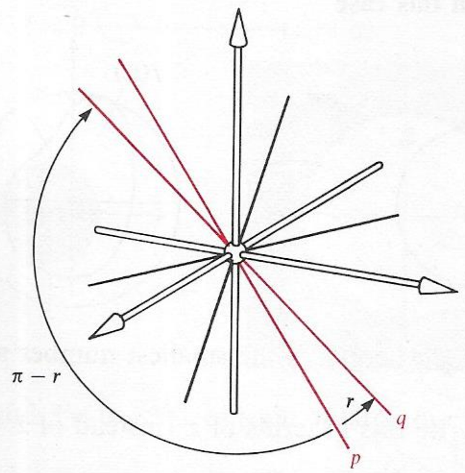
...because given any point z_0, with $|z_0| = s$, then for $r < \frac{1}{s}$, the set $\left\{z \in \mathbb{C} : |z| \ge \frac{1}{r}\right\}$ does not contain z_0. So there is no finite point contained in all the sets.

So the intersection of all the $D_{1/r}$, i.e. $\bigcap D_{1/r}$, contains the single point ∞.

RESTART THE TAPE

FRAME 11

Ray model of $\mathbb{RP}^2$: generalized distance



Define the distance between p and q, $\text{dist}(p, q)$ = angle r between p and q. (Take the smaller of the two angles between p and q, i.e. the one less than or equal to $\pi/2$.)

ME 12

Discs B_r in the ray model of $\mathbb{RP}^2$

Define $B_r = \{q : \text{dist}(p, q) \le r\}$.

CHECK that the B_r satisfy the three properties:

(1) defined for each real number $r > 0$;

(2) if $r > s$, then $B_r \supset B_s$;

(3) $\bigcap B_r = \{p\}$.

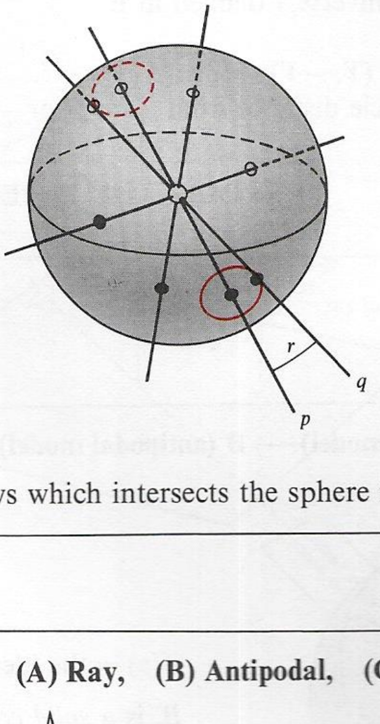
These are the properties of Frame 10.

Note that if $r > \pi/2$, then $B_r = B_{\pi/2}$, because $\pi/2$ is the largest angle you can have between two rays. (In fact, $B_{\pi/2} = \mathbb{RP}^2$.) However, this fact does not affect the question of whether or not the properties are satisfied.

RESTART THE TAPE

ME 13

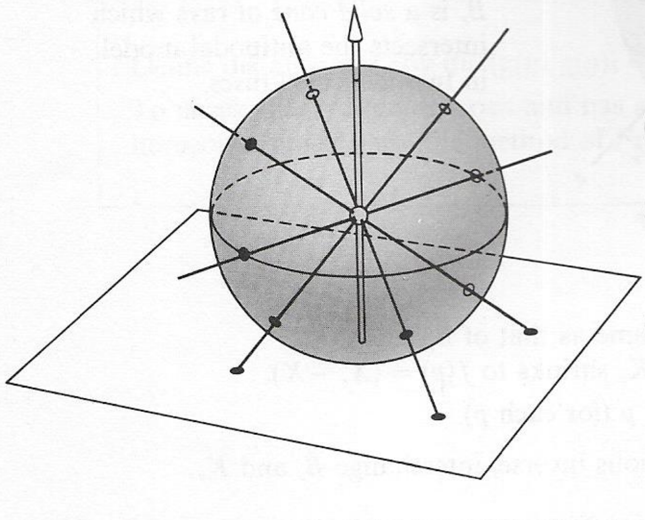
Intersections with the (unit) sphere make property (3) easy to see



B_r is a *solid cone* of rays which intersects the sphere in two discs.

ME 14

Three models of $\mathbb{RP}^2$: (A) Ray, (B) Antipodal, (C) Geometer's plane



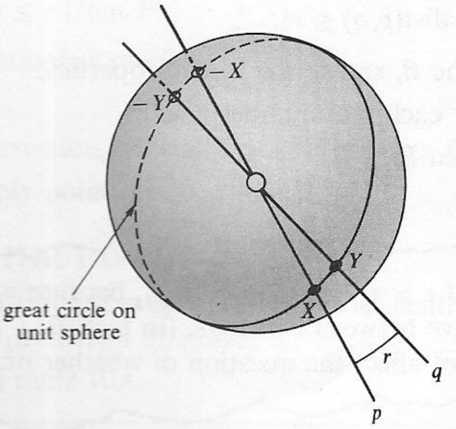
(A) Rays through the origin of $\mathbb{R}^3$.

(B) Sphere with antipodal pairs of points identified.

(C) Geometer's plane = $\mathbb{R}^2$ plus ideal points.

FRAME 15

Homeomorphism f: A (ray model) $\longrightarrow$ B (antipodal model)



Define $f(p) =$ the identified pair $(X, -X)$.

To check homeomorphism:

$$\left.\begin{array}{l} f \text{ one-to-one}\\ f \text{ onto}\end{array}\right\} \text{easy}$$

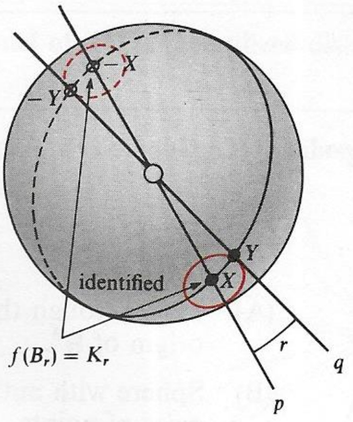
$$\left.\begin{array}{l} f \text{ continuous}\\ f \text{ has continuous inverse}\end{array}\right\} \begin{array}{l}\text{need distance}\\ \text{defined in B}\end{array}$$

Define $\operatorname{dist}((X,-X),(Y,-Y)) = \operatorname{dist}(p,q) = r$
(= shortest great circle distance from X to Y or $-Y$).

RESTART THE TAPE

FRAME 16

Proof that f: A (ray model) $\longrightarrow$ B (antipodal model) is a homeomorphism



$f(p) =$ the identified pair $(X, -X)$.

B_r is a *solid cone* of rays which intersects the antipodal model in two identified discs.

The radius of K_r is the same as that of B_r, i.e. r.
So as B_r shrinks to $\{p\}$, K_r shrinks to $f(p) = (X, -X)$.
Hence f is continuous at p (for each p).

To prove f has a continuous inverse, interchange B_r and K_r.

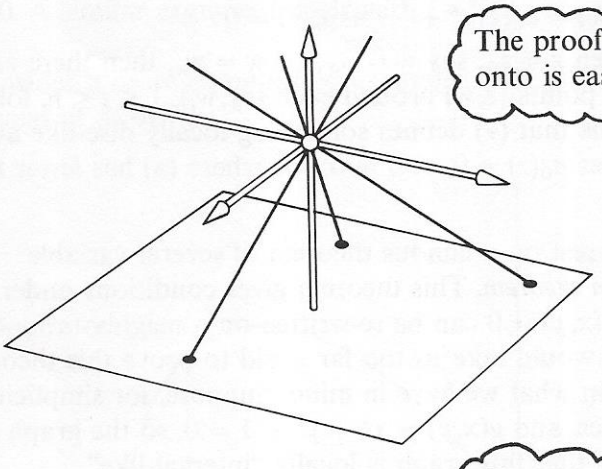
FRAME 17

Correspondence g: A (ray model) $\longrightarrow$ C (geometer's plane model)

(a) *For points of $\mathbb{R}^2$ in the plane model*

Homeomorphism check:

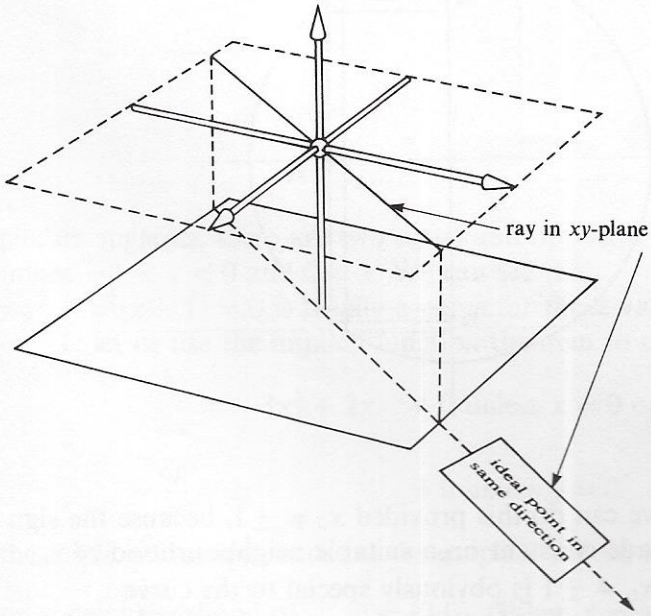
The proof that g is one-to-one and onto is easy.



Embedding of $\mathbb{R}^2$ in $\mathbb{R}^3$ (use plane $z = -1$)

To prove g is continuous with a continuous inverse, we need to define distance in C.

(b) *For the ideal points of the plane model*


ray in xy-plane corresponds to
ideal point in same direction

Define distance in C by $\text{dist}(g(p), g(q)) = \text{dist}(p, q)$.

To check that g is continuous and has a continuous inverse (and hence is a homeomorphism), use the method of Frame 15.

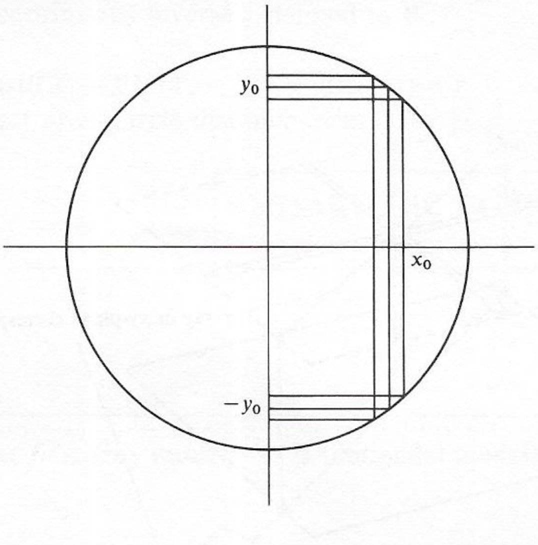
5.6 Appendix (Optional)

We sketch a proof that if

$$f(z, w) = a_0(z)w^n + \cdots + a_n(z) = 0 \qquad (*)$$

has n distinct roots in w when $z = z_0$, say $w = w_1, \ldots, w = w_n$, then there are disc-like neighbourhoods of points (z, w) around each (z_0, w_i), $1 \leq i \leq n$, for which $f(z, w) = 0$. This means that $(*)$ defines something locally disc-like away from points z for which either $a_0(z) = 0$, or $z = \infty$, or where $(*)$ has fewer than n roots.

A rigorous argument would rest on a famous theorem of several variable calculus: the *implicit function theorem.* This theorem gives conditions under which an implicit function $g(x, y) = 0$ can be re-written on a neighbourhood in the x-domain as $y = h(x)$. It would take us too far afield to prove this theorem, but some pictures will explain what we have in mind. Suppose, for simplicity, that x and y are real variables, and $g(x, y) = x^2 + y^2 - 1 = 0$, so the graph of g is a circle. We want to check that this graph is locally "interval-like". Algebraically, we want to write $y = \pm(1 - x^2)^{\frac{1}{2}}$ and to interpret this as two functions defined near x_0, one for each sign of the square root.



Pictorially, we find that we can do this provided $x_0 \neq \pm 1$, because the sign of the square root can be made constant on a suitable neighbourhood of x_0, $x_0 \neq \pm 1$. The condition $x_0 \neq \pm 1$ is obviously special to the curve $x^2 + y^2 - 1 = 0$, and in the implicit function theorem it is formulated for a general g as $\dfrac{\partial g}{\partial y} \neq 0$. It is easy to see why this condition is relevant geometrically. When $x_0 \neq \pm 1$, consider either of the points $(x_0, \pm y_0)$ on the circle, and observe that the tangent to the circle at (x_0, y_0) is not vertical. Assuming that the tangent is a good approximation to the circle on some neighbourhood, we see that the function g is approximately given by the linear implicit function

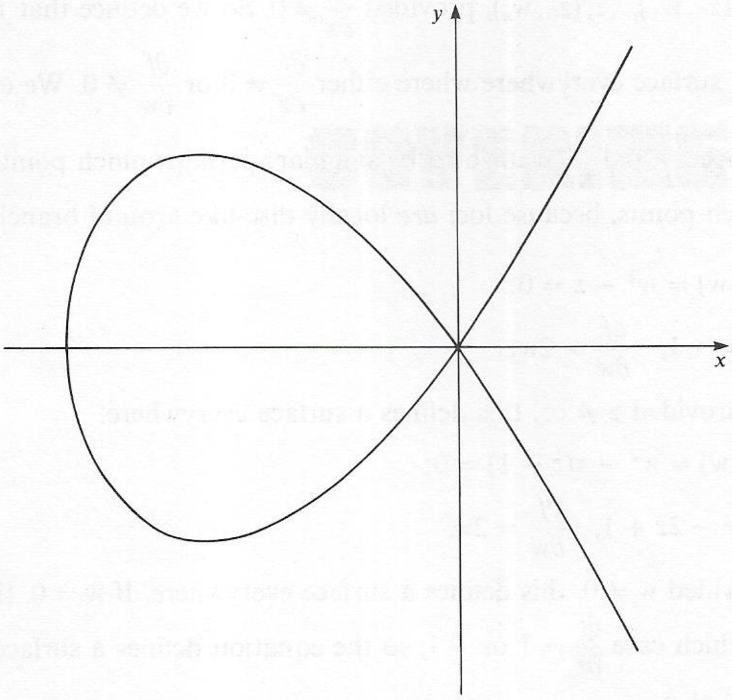
$$y - y_0 = m(x - x_0), \quad \text{where } m = -\left(\frac{\partial g}{\partial x}\Big/\frac{\partial g}{\partial y}\right),$$

the partial derivatives being evaluated at (x_0, y_0).

The implicit function theorem says that if $g(x, y) = 0$ and $\dfrac{\partial g}{\partial y} \neq 0$ on some neighbourhood D of x_0, then there is a differentiable function $h: \mathbb{R} \longrightarrow \mathbb{R}$ such that $y = h(x)$ on D. That the image of D on the circle $g(x, y) = 0$ is an interval means that $g = 0$ defines a *curve* near that point.

Remark By exchanging the roles of x and y, we can use the implicit function theorem to show that the circle is locally a curve near $(1,0)$, because $\left.\dfrac{\partial g}{\partial x}\right|_{(1,0)} = 2 \neq 0$, so the theorem yields a function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$, such that $x = h(y)$ on a neighbourhood of $y_0 = 0$. A similar argument deals with $(-1,0)$.

Now consider a more complicated example, such as $y^2 = x^2(x-1)$, which has a *double point* at $(0,0)$.



As the picture suggests, there are two equal and opposite values of y for each x in the ranges $-1 < x < 0$ and $0 < x$. We can see that $g(x,y) = y^2 - x^2(x-1) = 0$ is locally a curve for those values of x, and also near $x = -1$; let us use the implicit function theorem to confirm this.

$$\frac{\partial g}{\partial x} = -3x^2 + 2x \qquad \neq 0 \text{ unless } x = 0 \text{ or } \tfrac{2}{3};$$

$$\frac{\partial g}{\partial y} = 2y \qquad \neq 0 \text{ unless } y = 0.$$

So, by the implicit function theorem, the graph is indeed locally interval-like, except on a neighbourhood of $(x,y) = (0,0)$. (When $x = \frac{2}{3}$, $y \neq 0$; so $\dfrac{\partial g}{\partial y} \neq 0$ and we use the implicit function theorem to write $x = h(y)$. That $\dfrac{\partial g}{\partial x} = 0$ when $x = \frac{2}{3}$ does not then matter.)

However, when $x = y = 0$, $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ both vanish, and the implicit function theorem cannot help. In fact, the graph makes it clear that $g = 0$ is not locally interval-like here — it represents a one-dimensional pinch point, if you like.

We shall now apply these ideas to the equation $f(z,w) = 0$ and think of z and w as variables in $\mathbb{C}$.

Since $f(z_0, w)$ has n distinct roots, by assumption, and f is certainly continuous in z and w, it is possible to restrict z to a disc D around z_0 for which $f(z,w) = 0$ has n distinct roots. This means that whenever $z \in \dot{D}$, $f(z,w) = 0$ never has a repeated root, whence $f(z, w) = 0$ and $\dfrac{\partial f}{\partial w}(z, w) = 0$ never have a root in common. So if $f(z_1, w_1) = 0$, then $\dfrac{\partial f}{\partial w}(z_1, w_1) \neq 0$, i.e. $\dfrac{\partial f}{\partial w}$ never vanishes on D. So the implicit function theorem applies to each of the patches around $(z_0, w_1), \ldots, (z_0, w_n)$, and we deduce that each of them is a piece of surface. The same argument would hold if we used z in place of w, and the implicit function theorem would then map a small disc in the w-domain onto each neighbourhood of $(z_0, w_1), \ldots, (z_0, w_n)$, provided $\dfrac{\partial f}{\partial z} \neq 0$. So we deduce that $f(z, w) = 0$ defines a surface everywhere where either $\dfrac{\partial f}{\partial z} \neq 0$ or $\dfrac{\partial f}{\partial w} \neq 0$. We expect points where both $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial w}$ vanish to be singular; perhaps pinch points or worse, but *not* branch points, because loci *are* locally disc-like around branch points.

EXAMPLE (i) $f(z, w) = w^2 - z = 0$:

$$\frac{\partial f}{\partial z} = -1, \quad \frac{\partial f}{\partial w} = 2w;$$

so provided $z \neq \infty$, this defines a surface everywhere.

(ii) $f(z, w) = w^2 - z(z - 1) = 0$;

$$\frac{\partial f}{\partial z} = -2z + 1, \quad \frac{\partial f}{\partial w} = 2w.$$

Provided $w \neq 0$, this defines a surface everywhere. If $w = 0$, then $z = 0$ or 1, in which case $\dfrac{\partial f}{\partial z} = 1$ or -1, so the equation defines a surface at $(0, 0)$ and $(1, 0)$ also.

This method extends to infinite points as well, if the algebraic method of compactification is used. Thus $(z, w) = (\infty, \infty)$ is the only infinite point on $f(z, w) = w^2 - z$. It corresponds to $(\tilde{z}, \tilde{w}) = (0, 0)$ on $f(\tilde{z}, \tilde{w}) = \tilde{w}^2 - \tilde{z} = 0$, and since $\dfrac{\partial f}{\partial \tilde{z}} = -1 \neq 0$, the locus is a surface there too.

EXAMPLE $f(z, w) = w^2 - z(z - 1) = 0$ at infinity.

The algebraic compactification gives

$$f(\tilde{z}, \tilde{w}) = \tilde{z}^2 - \tilde{w}^2(1 - \tilde{z}) = 0;$$

$$\frac{\partial f}{\partial \tilde{z}} = 2\tilde{z} + \tilde{w}^2 = 0 \text{ at } (\tilde{z}, \tilde{w}) = (0, 0),$$

$$\frac{\partial f}{\partial \tilde{w}} = -2\tilde{w}(1 - \tilde{z}) = 0 \text{ at } (\tilde{z}, \tilde{w}) = (0, 0).$$

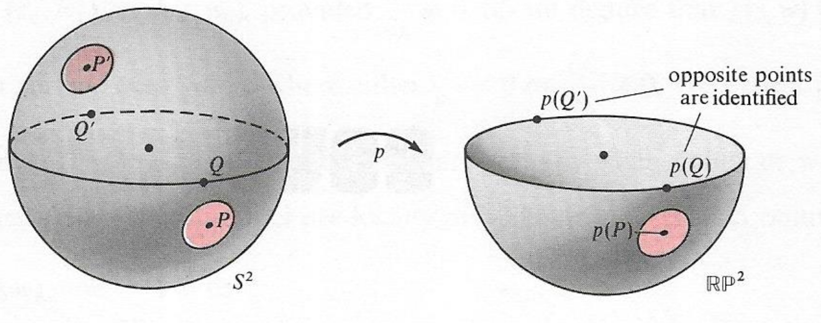
So $(z, w) = (\infty, \infty)$ appears to be a singular point on the locus. We saw earlier that this is so.

UNIT 6

COVERING SPACES

6.1 Covering Spaces

We first met an example of a covering space in *Unit 3*, where we showed that the real projective plane $\mathbb{RP}^2$ can be obtained from the sphere S^2 by identifying antipodally opposite points. Starting with the standard unit sphere in $\mathbb{R}^3$, namely $S^2 = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$, we identify (x, y, z) with $(-x, -y, -z)$.



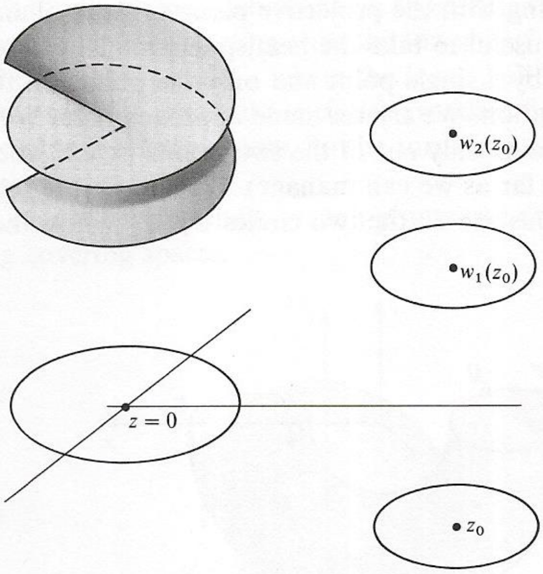
If we represent $\mathbb{RP}^2$ as the southern hemisphere, with opposite points on the equator identified, then we can define the map $p: S^2 \longrightarrow \mathbb{RP}^2$ as follows. A point P in the southern hemisphere is left fixed; a point P in the northern hemisphere is sent to the diametrically opposite point $P'(= -P)$ in the southern hemisphere representing $\mathbb{RP}^2$. For points Q on the equator, p sends Q to the pair of opposite points $\{\pm Q\}$ (identified, of course). Every point of $\mathbb{RP}^2$ is the image of two points of S^2 under p. A small disc round $p(P)$ is homeomorphic to a small disc round P, *and* to another round $P' = -P$; so the whole surface $\mathbb{RP}^2$ is "covered" twice by the map p. Two points of S^2 correspond under p to just one on $\mathbb{RP}^2$. This is clear from the diagram, for all points except possibly for those on the equator. There, we have to remember that we are identifying a point, such as Q, with its opposite point, $-Q$, to a single point; so this single identified point of $\mathbb{RP}^2$ is the image of both Q and $-Q$ under p, and so is not an exception.

This, then, is an example of a *covering space*. For a surface S (in our case $\mathbb{RP}^2$), there is given another space $\tilde{S}$ (in our case S^2) and a map $p: \tilde{S} \longrightarrow S$ such that the inverse image of any $x \in S$ under p has exactly n points in $\tilde{S}$. If every point in S has n "copies" in $\tilde{S}$, the covering is said to be an *n-fold covering*. (There is a further technical condition which we shall deal with later.)

So the map $S^2 \longrightarrow \mathbb{RP}^2$ discussed above gives an example of a 2-fold covering of the projective plane. We have met another example which *nearly* satisfies the covering space condition.

Consider the locus of $w^2 = z$. In *Unit 5* we saw that this is a Riemann sphere, $\mathbb{C} \cup \{\infty\}$, and that the map $p: (z, w) \longmapsto z$ covers each point of the z-sphere with two points of the locus — *except* for $z = 0$ and $z = \infty$. $(0, 0)$ is the only point over $z = 0$, and (∞, ∞) is the only point over $z = \infty$ (these are branch points of the locus).

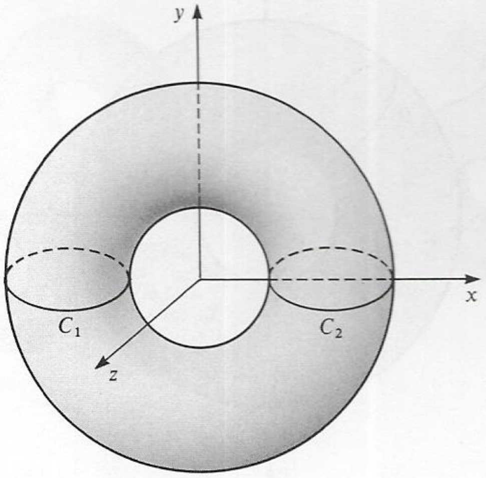
We pictured this locus as two copies of the little disc round each point z_0, sitting over the little disc in the z-plane, except for $z_0 = 0$ and $z_0 = \infty$, where we had to take a more complicated picture. The function p can then be imagined in terms of this picture as mapping each point (z, w) in the locus to the point z in the z-plane lying immediately below it. This is clearly similar to the 2-fold coverings we have just defined, except at the "bad" branch points 0 and ∞; we call this a *branched 2-fold cover*.



These two situations are very general in topology, and **B** deals with them in his Section 3.2. In the above example, the surface S (the image of the covering map p) is the sphere S^2, but we are not restricted to covering the sphere; moreover, as we shall see, many surfaces may play the role of the covering space, $\tilde{S}$, which is mapped onto S.

Before we give more precise definitions, let us look at some more examples. The covering of the projective plane by the sphere came from a two-fold *symmetry* of the sphere: the central symmetry, sending each point to its opposite point. We can find many other examples of covering spaces related to symmetries of surfaces.

EXAMPLE 1 Take $\tilde{S}$ to be the torus, symmetrically situated round the origin in $\mathbb{R}^3$. Verify that $(x, y, z) \longmapsto (-x, -y, -z)$ maps the torus to itself (it is, in fact, a central symmetry). Form a new surface, by identifying each point (x, y, z) with the "opposite" point $(-x, -y, -z)$, and show that the resulting surface is a Klein bottle.

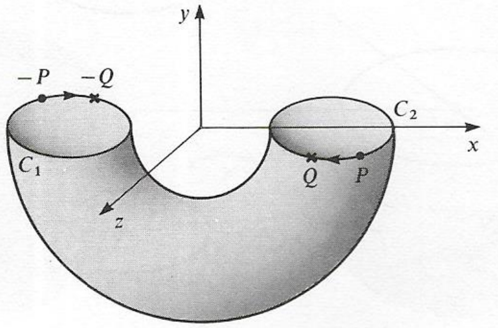


$$z^2 = (3 - \sqrt{(x^2 + y^2)})(\sqrt{(x^2 + y^2)} - 1)$$

Solution

We have taken the torus as described in Chapter 1 of **B** for a definition, though it does not really affect the argument. It does, however, make it very obvious that if (x, y, z) is on the torus, so is $(-x, -y, -z)$.

Now, when we were dealing with the projective plane as pairs of antipodal points of S^2, we found it useful to take the hemisphere model where *most* points of $\mathbb{RP}^2$ were represented by a single point and only the points on the equator were involved in identification. We try the same approach here. We look for a subset S' of $\tilde{S}$ which contains only *one* of the two points (x, y, z) and $(-x, -y, -z)$ (at least as far as we can manage). We choose the region $y \leq 0$; then the only identifications are on the two circles C_1, C_2 where the plane $y = 0$ cuts the torus.

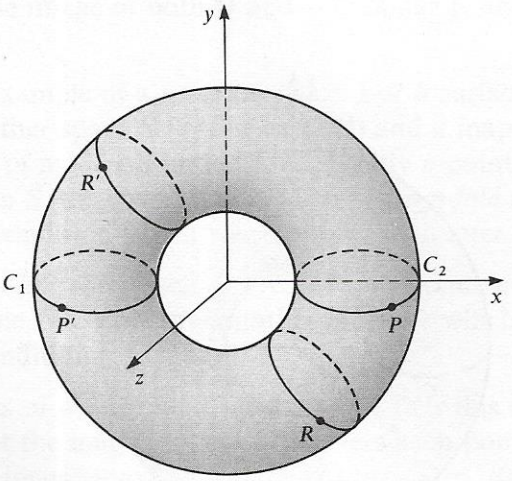


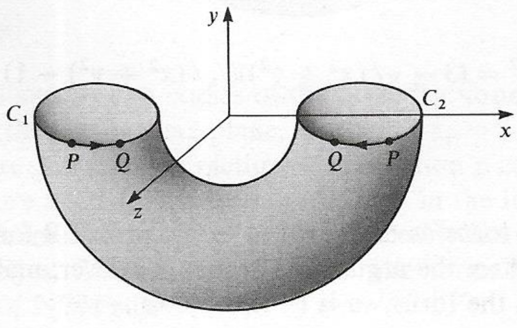
Each point P on C_2 is identified with $-P$ on C_1. If we look at a second point, Q, we can see the direction in which the two circles have to be identified. So the result of identifying pairs on the torus is the same as identifying the two circular ends of the half torus in the above figure, in the sense given by the arrows. It is not hard to see that this is, in fact, a Klein bottle (not another torus). So we have shown that there is a map $T^2 \longrightarrow K$ which makes the torus a 2-fold cover of the Klein bottle.

EXAMPLE 2 With the same torus as in Example 1, identify each point with its image under a rotation through π about the z-axis. What is the resulting space?

Solution

We can adopt the same procedure as before. Every point on the torus not in the lower half is rotated into the lower half by the rotation through π.

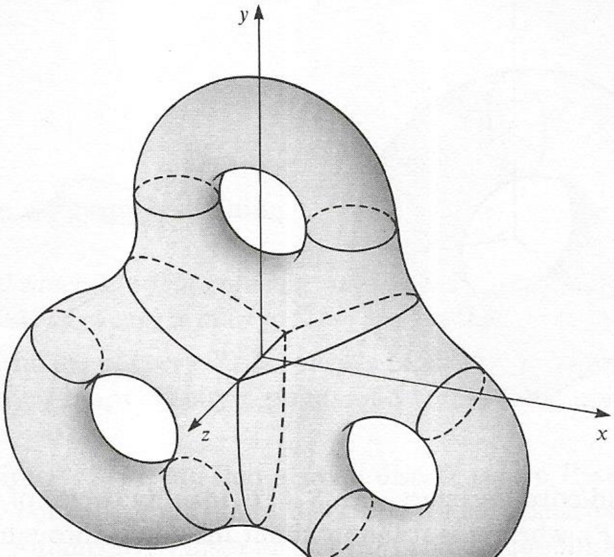




So the result of identifying each point on the torus with its image after rotation through π is the same as starting with the lower half torus and identifying points on the circles C_1 and C_2 where the lower half torus meets the upper half. Putting in the sense in which these circles are identified, as before, we see that this time we obtain another torus. So the torus is a double cover of itself! ☐

In the next example we see how branch points can arise from this method of constructing covering spaces.

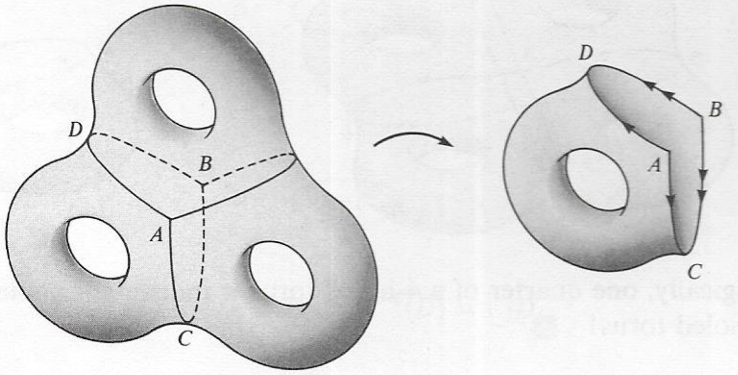
EXAMPLE 3



$\tilde{S}$ is a 3-holed torus symmetrically situated in $\mathbb{R}^3$ so that rotation through $2\pi/3$ about the z-axis carries $\tilde{S}$ into itself. Identify the points P, $g(P)$ and $g(g(P))$, where g is rotation through $2\pi/3$, for each point P of $\tilde{S}$. What is the resulting surface?

Solution

Consider S', which is "one-third" of $\tilde{S}$.



Points on the arcs AD, AC correspond after rotation, as do points on BD and BC, and the identifications are as shown. Each other point of $\tilde{S}$ not in S' corresponds to exactly one point of S', either after rotation through $2\pi/3$ or after rotation through twice $2\pi/3$. So the surface S obtained after identifications is the same as S' with the identifications on the four arcs, which is again a torus.

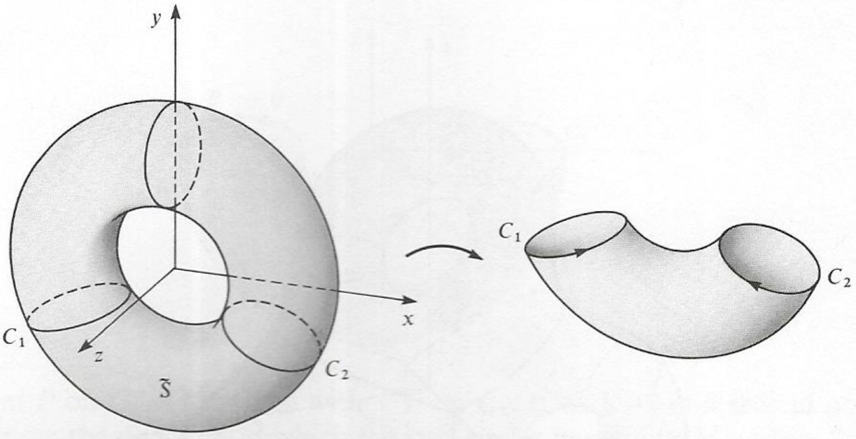
Notice that this time there are three points of $\tilde{S}$ over each point of S, *except* for the points A and B (which are on the axis of rotation of g). So this is a branched covering, because A and B are branch points.

SAQ 1 Show, by means of suitable diagrams and functions, how Example 2 can be adapted to give a torus as an n-fold covering space of a torus.

SAQ 2 Similarly, adapt Example 3 to show how an n-holed torus is an n-fold branched cover of an ordinary torus.

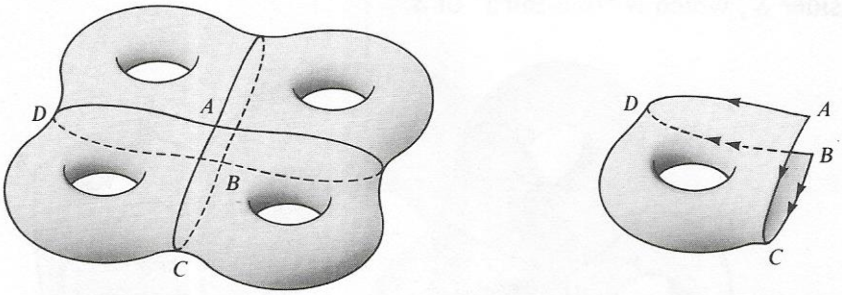
SAQ 3 Using the ideas of these problems, show that a sphere can be an n-fold branched covering space of a sphere.

SOLUTION TO SAQ 1

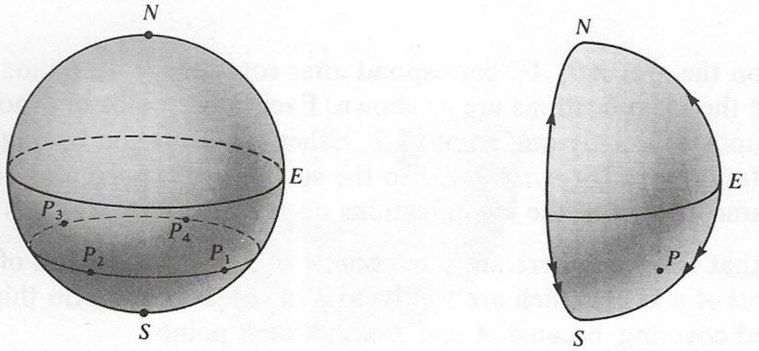


To get a 3-fold covering, start with $\tilde{S} = T$. Identify triples of points on the torus: P, $g(P)$, $g \circ g(P)$, where g is rotation about the z-axis through $2\pi/3$. Every triple of points related in this way contains at least one point in the "lowest third", S'. For points on the circles C_1, C_2, C_3, dividing $\tilde{S}$ into three curved cylinders, there are two points in S', one in C_1 and one in C_2, which have, therefore, to be identified. This gives a torus. The argument for an n-fold covering is similar. ■

SOLUTION TO SAQ 2 Exactly the same argument works for an n-holed torus as we gave in Example 3. The picture below is for a 4-holed torus.



Topologically, one quarter of a 4-holed torus is indistinguishable from one third of a 3-holed torus! ■



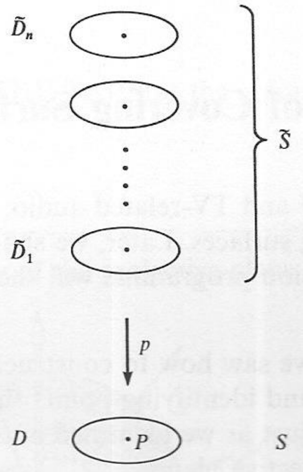
To obtain (for example) a 4-fold branched cover of the sphere, start with the unit sphere and identify all the four points obtained from P and its rotations through $\pi/2$, $2\pi/2$, $3\pi/2$ about the N–S axis. Every set of four points obtained in this way from a single point P has one point in the "front right-hand quarter" of S^2, as shown; points on the edges are identified, after rotation by $\pi/2$, as shown by the arrows. The result of identifying points in fours on S^2 is, therefore, the result of identifying the edges of the quarter-sphere, which is again a sphere. N and S are branch points. The general case is similar, dividing S^2 into n segments and identifying points n at a time. ■

It is time to give a rigorous definition.

DEFINITION A *regular*, or *unbranched n-fold covering surface* is a surface $\tilde{S}$, another surface S (called the *base surface*) and a map $p: \tilde{S} \longrightarrow S$ such that:

(i) the inverse image of every $P \in S$ consists of exactly n points of $\tilde{S}$ — put another way, f maps exactly n points of $\tilde{S}$ to the same point P of S, for each P;

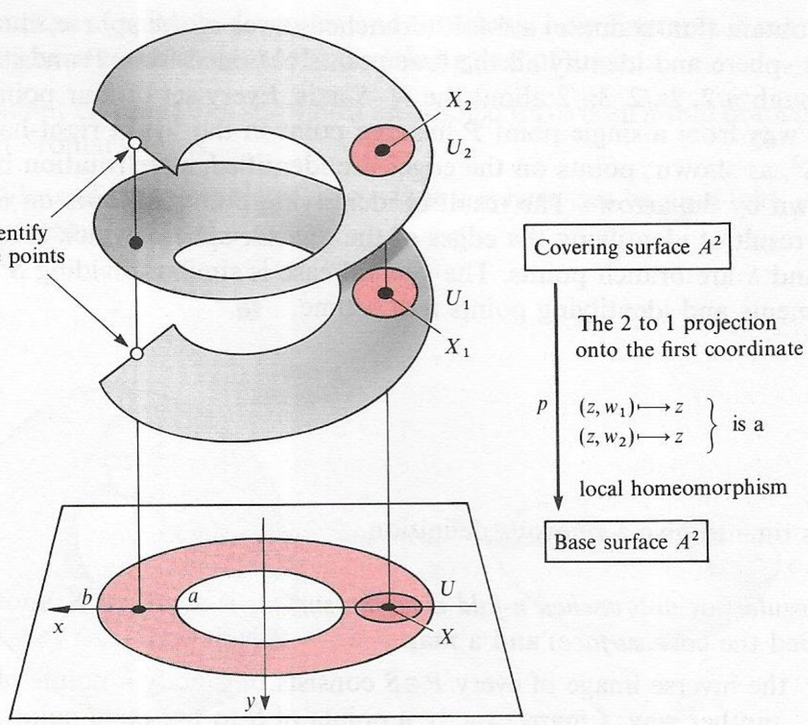
(ii) for any point $P \in S$ we can find a small disc D, centre P, so that the inverse of D consists of n disjoint discs in $\tilde{S}$ and P restricted to any of these discs $\tilde{D}_i$ is a homeomorphism of $\tilde{D}_i$ with D. (This condition is sometimes expressed by saying that p is a *local homeomorphism*: given a point $\tilde{P}$ with $p(\tilde{P}) = P$ then p maps a patch round $\tilde{P}$ homeomorphically to a patch round P.)



$$\tilde{D}_i \cong D \qquad p^{-1}(D) = \bigcup \tilde{D}_i$$

We have seen the same kind of picture when considering loci in *Unit 5*.

EXAMPLE 4 Recall the locus $w^2 = z$. For each point z, if w_1 and w_2 are its two square roots, we represent the corresponding points (z, w_1), (z, w_2) of the locus as two points sitting vertically above z. Now consider the part of this locus corresponding to an annulus $(a \leq |z| \leq b)$ in the z-plane. This part is an annulus A^2 which double covers the z-annulus.


X_2
U_2
identify
edge points
Covering surface A^2
U_1
X_1
The 2 to 1 projection
onto the first coordinate
p
$(z, w_1) \longmapsto z$
$(z, w_2) \longmapsto z$
is a
local homeomorphism
Base surface A^2
U
b
a
x
X
y

TERMINOLOGY If we restrict attention to the disc U_1 and consider the *inverse* of p, then this inverse is said to "lift" the disc U to U_1. Similarly for U_2.

Thus, for a regular n-fold covering, a small disc in the base surface lifts to n small discs in the covering surface.

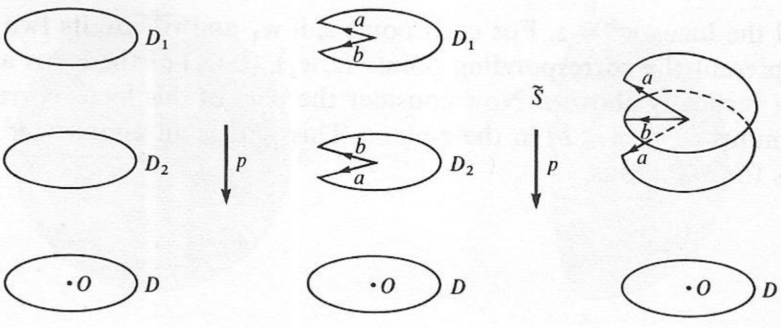
The covering surface $\tilde{S}$ is sometimes called the *total space* of the covering.

6.2 Some Properties of Covering Surfaces

This section contains TV and TV-related audio. We begin with another method for constructing covering surfaces. Later, we shall look further at branched coverings, and the television programme will show many other examples of covering surfaces.

In the previous section, we saw how to construct covering spaces by considering symmetries of a surface and identifying points that correspond to one another under these symmetries, just as we identified antipodally opposite points on the sphere to obtain the projective plane.

Another method of construction is to start with n copies of a given surface and join these together in a suitable way. An example would be to start with a disc and take two copies of it, as follows.


D_1
D_2
p
D
O
a
b
$\tilde{S}$

Then cut the two copies along a radius — it does not really matter which radius we choose. Now identify the edges of the cuts as shown to form $\tilde{S}$. We cannot join both edges in $\mathbb{R}^3$, but you will recognize the picture as that of a branch point (shown in *Unit* 5). From the construction, it is evident that this is *nearly* a regular covering space (as defined at the end of the previous section), but has the one point — the branch point — where condition (i) fails. There is only a single point in the covering space $\tilde{S}$ which maps to O under p. (Of course, condition (ii) fails as well.) This is an example of a *branched covering*.

We shall be dealing with other examples of branched coverings in this section and the next, but we shall not give a definition of a general branched covering space. Roughly, a branched covering surface is an n-fold covering surface $p: \tilde{S} \longrightarrow S$, except that at a finite number of points, $P_1, \ldots, P_r$, the inverse image $p^{-1}(P_i)$ has fewer than n points. The points $p^{-1}(P_i)$ are called the *branch points* of the covering surface. Of course, to understand what the surface *looks* like, we have to know how it looks at each branch point, and the next section will give some detailed examples for the case when S is a sphere. Branched coverings are very useful: every orientable surface (without boundary) is a branched cover of the sphere; and, more surprisingly, the analogue of this is also true in three dimensions, though three-dimensional manifolds are very much more complicated than surfaces.

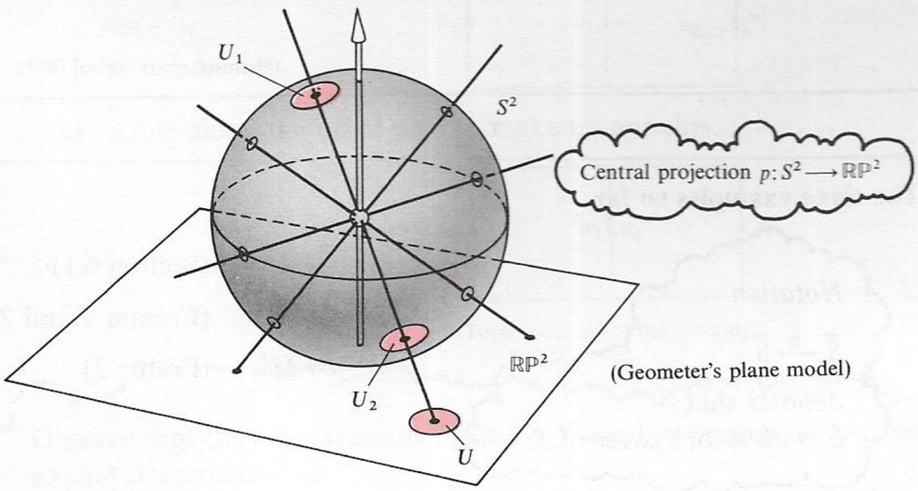
The accompanying television programme shows that several situations we have met are, indeed, examples of covering surfaces.

Now is a good time to watch TV6.

TV Overview

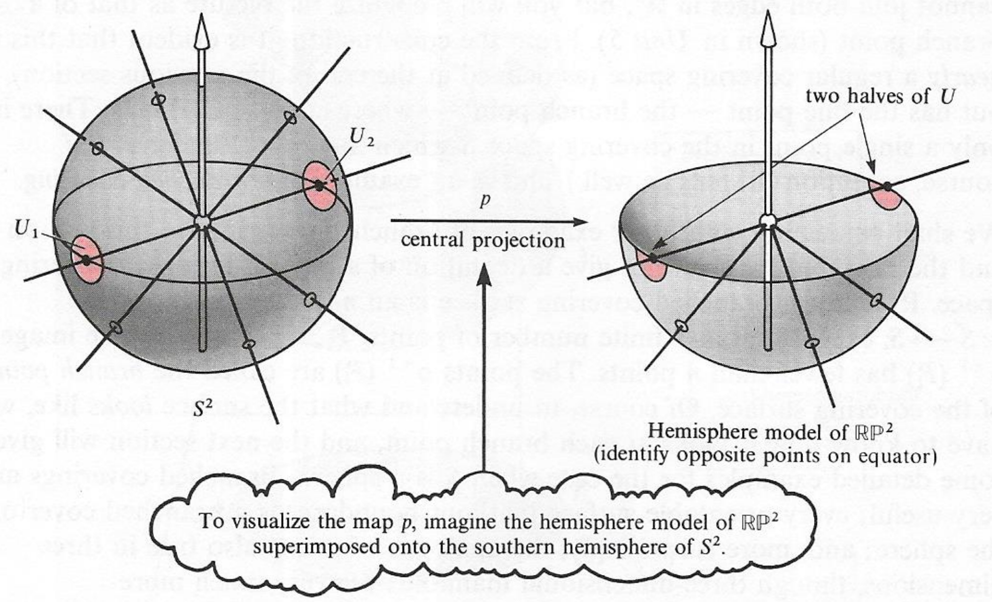
After watching TV6, START THE TAPE at the second section of Audio Cassette A2.

FRAME 1 **The sphere as a double cover of the projective plane**



FRAME 2

The patches of S^2 mapping to a patch around an ideal point of $\mathbb{RP}^2$


U_2
U_1
p
central projection
two halves of U
S^2
Hemisphere model of $\mathbb{RP}^2$
(identify opposite points on equator)

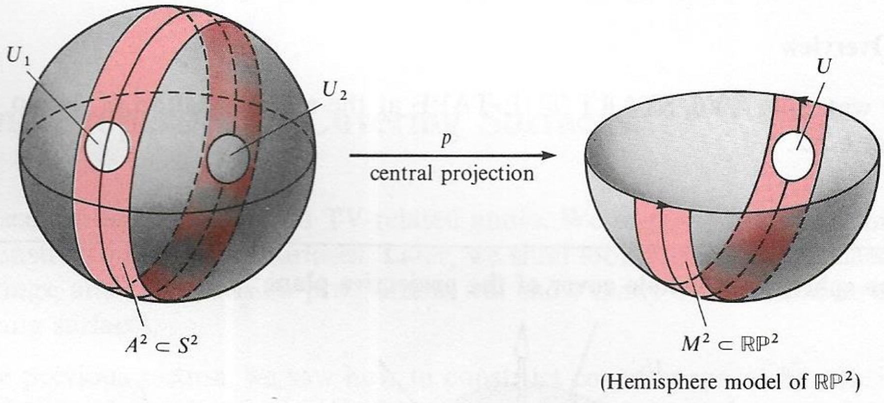
To visualize the map p, imagine the hemisphere model of $\mathbb{RP}^2$ superimposed onto the southern hemisphere of S^2.

The bottom half of U_1 maps to the left half of U in $\mathbb{RP}^2$, and the top half of U_1 maps to the right half of U (and vice-versa for U_2).

RESTART THE TAPE

FRAME 3

A cylindrical band in S^2 double covers a Möbius band in $\mathbb{RP}^2$


U_1
U_2
p
central projection
U
$A^2 \subset S^2$
$M^2 \subset \mathbb{RP}^2$
(Hemisphere model of $\mathbb{RP}^2$)

FRAME 4

The three examples so far

Notation

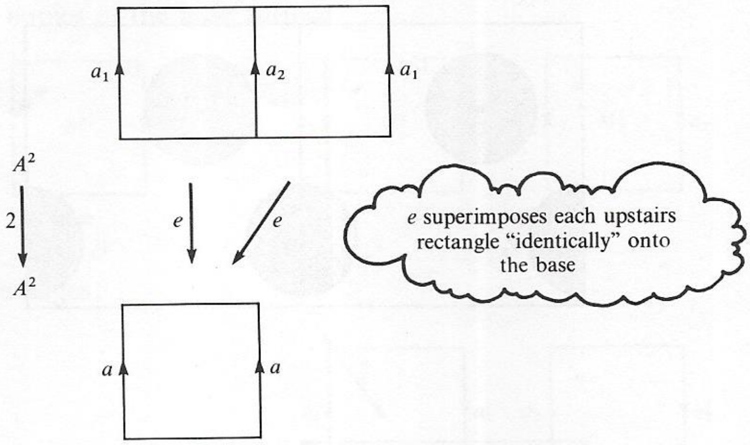
$\tilde{S} \xrightarrow{n} \tilde{S}$

denotes that
S is an n-fold cover of S.

$A^2 \xrightarrow{2} A^2$	(Section 6.1)
$S^2 \xrightarrow{2} \mathbb{RP}^2$	(Frames 1 and 2)
$A^2 \xrightarrow{2} M^2$	(Frame 3)

FRAME 5

$A^2 \xrightarrow{2} A^2$ in terms of polygons (TV treatment)

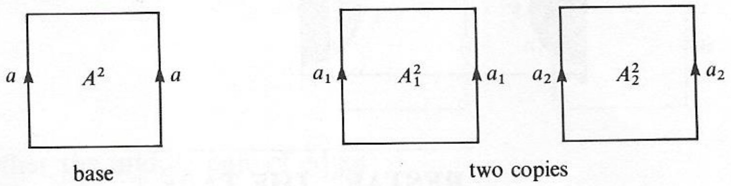

a_1
a_2
a_1
A^2
2
A^2
e
e
a
a

e superimposes each upstairs rectangle "identically" onto the base

FRAME 6

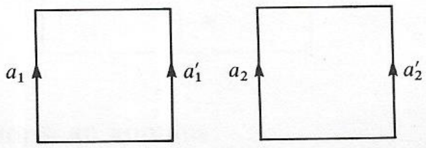
$A^2 \xrightarrow{2} A^2$ in terms of polygons (alternative treatment)

(i) Make two copies of the base surface:

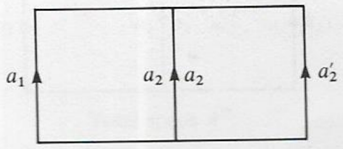

a
A^2
a
base
a_1
A_1^2
a_1
a_2
A_2^2
a_2
two copies

(ii) Glue the copies to make an annulus — in three steps.

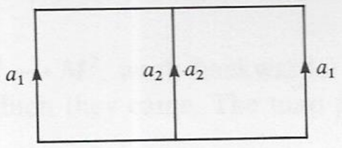
(a) Cut up the edges to make two rectangles:



(b) Glue along one pair of edges to make a big rectangle:



(c) Glue along the other pair to make an annulus:



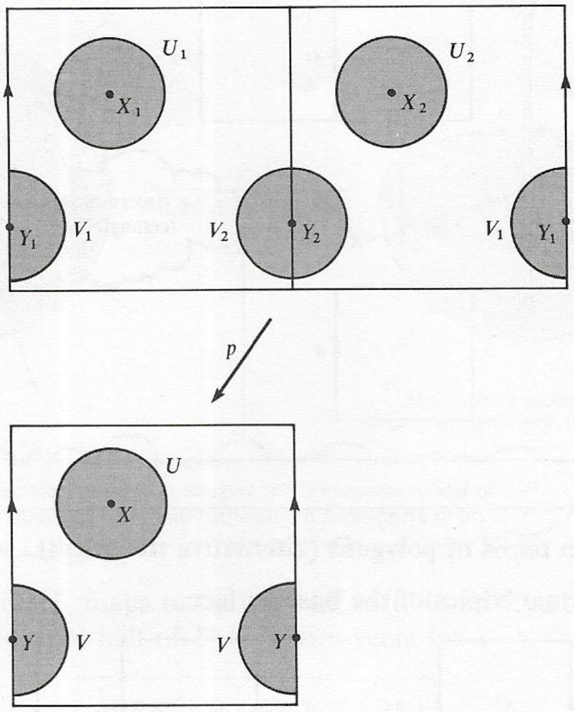
Total space A^2 of the covering

Observe that the edge symbols a_1 and a_2 have been permuted on their second occurrences, i.e.

in the first diagram, the order is $a_1, a_1, a_2, a_2,$

whereas in the last one, the order is a_1, a_2, a_2, a_1.

FRAME 7 The map $p: A^2 \xrightarrow{2} A^2$ is a local homeomorphism

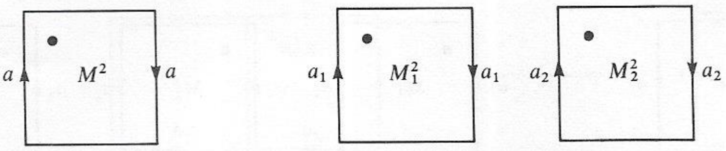

U_1
X_1
U_2
X_2
Y_1
V_1
V_2
Y_2
V_1
Y_1
p
U
X
Y
V
V
Y

RESTART THE TAPE

FRAME 8

$A^2 \xrightarrow{2} M^2$ in terms of polygons (Frame 6 treatment)

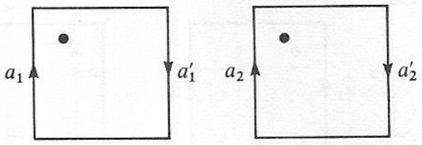
Make two copies of the base surface:


a
M^2
a
a_1
M_1^2
a_1
a_2
M_2^2
a_2

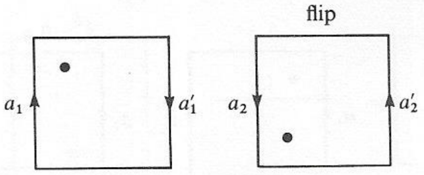
base

two copies

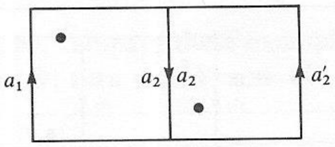
Unglue the edges:



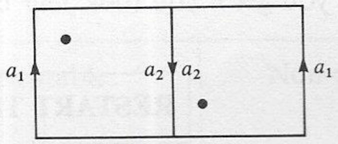
Give the second copy a half twist (i.e. flip it, top to bottom):


flip
a_1
a_1'
a_2
a_2'

Glue together the middle pair of edges:



Glue together the "other" pair to form an annulus:



Total space A^2

Notice that (as in Frame 6) the edge symbols have been permuted on their second occurrences.

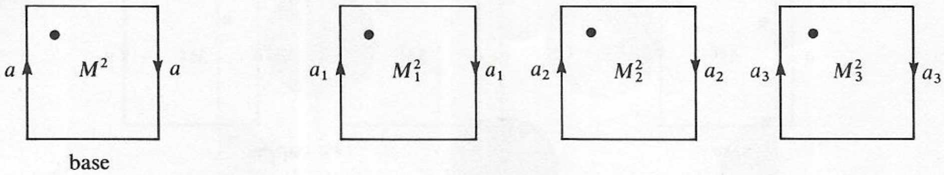
To obtain the projection map, $p\colon A^2 \longrightarrow M^2$, work backwards: i.e. map the points of A^2 to those in M^2 from which they came. The map p is a local homeomorphism.

RESTART THE TAPE

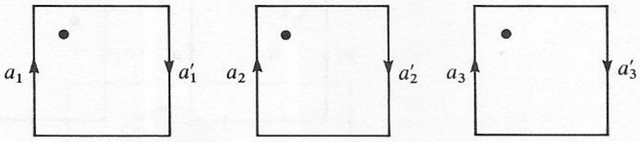
FRAME 9

$M^2 \xrightarrow{3} M^2$

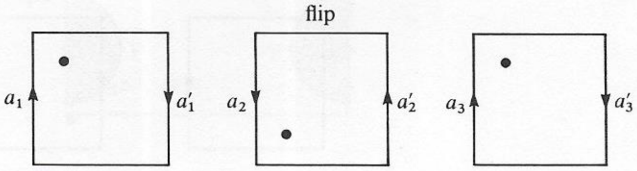
Make three copies of the base surface:


a
M^2
a
base
a_1
M_1^2
a_1
a_2
M_2^2
a_2
a_3
M_3^2
a_3

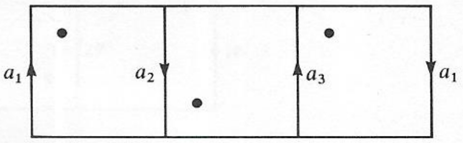
Unglue their edges:


a_1
a_1'
a_2
a_2'
a_3
a_3'

Flip the middle copy over:


flip
a_1
a_1'
a_2
a_2'
a_3
a_3'

Glue together the two middle pairs of edges and the outer pair:



Total space M^2

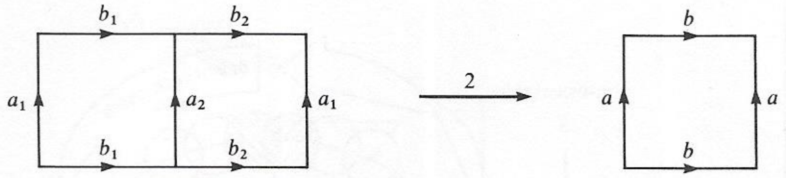
What would you get if you took *four* copies of the base? And so on?

RESTART THE TAPE

FRAME 10

Identify the unlabelled edges in Frames 6, 8 and 9

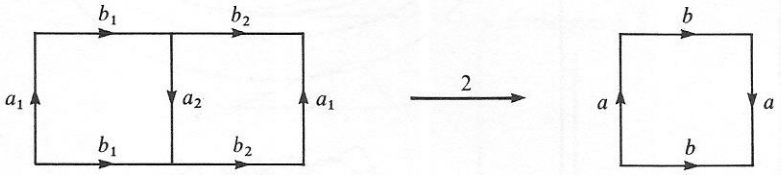
$T^2 \xrightarrow{2} T^2$



Total space T^2

base T^2

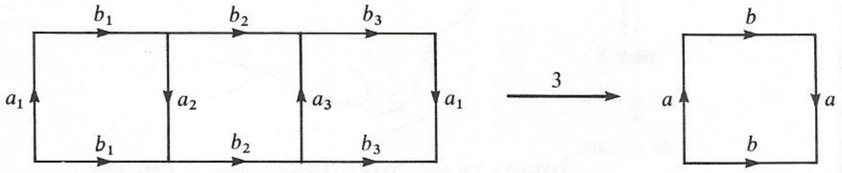
$T^2 \xrightarrow{2} K^2$



Total space T^2

base K^2
(Klein bottle)

$K^2 \xrightarrow{3} K^2$



Total space K^2

base K^2

Before restarting the tape, compare these examples with Frames 6, 8 and 9; then read Frame 11.

FRAME 11

Examples $\tilde{S} \xrightarrow{n} S$ so far

Orientable ↓ Orientable	Orientable ↓ Non-orientable	Non-orientable ↓ Non-orientable
$A^2 \xrightarrow{2} A^2$ (Frames 5, 6, 7)	$S^2 \xrightarrow{2} \mathbb{RP}^2$ (Frames 1, 2) $A^2 \xrightarrow{2} M^2$ (Frames 3, 8)	$M^2 \xrightarrow{3} M^2$ (Frame 9)
$T^2 \xrightarrow{2} T^2$ (Frame 10)	$T^2 \xrightarrow{2} K^2$ (Frame 10)	$K^2 \xrightarrow{3} K^2$ (Frame 10)

Make sure you understand these examples. (Read through the relevant frames, if necessary.)

FRAME 12

Theorem A A non-orientable surface cannot cover an orientable one

Orientation-reversing circle in S.

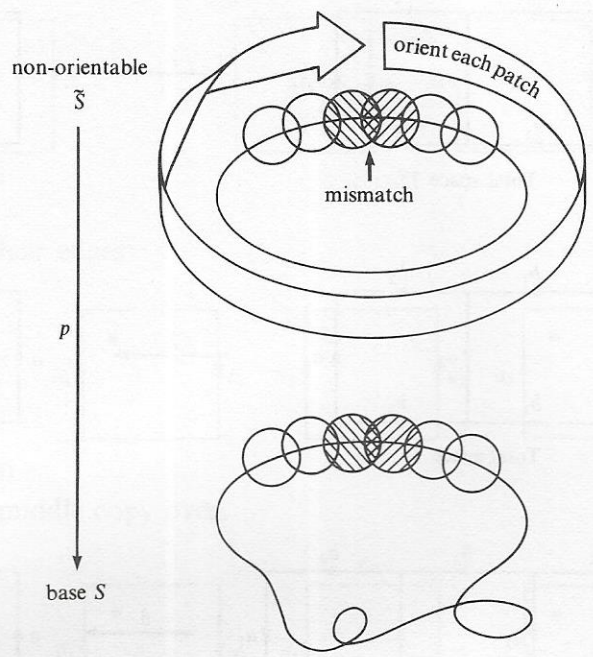


Image of orientation-reversing circle:
must also reverse orientation, because
p is a local homeomorphism.

Hence we *cannot* have, say, $M^2 \xrightarrow{2} A^2$, or $\mathbb{RP}^2 \xrightarrow{2} S^2$.

RESTART THE TAPE

FRAME 13

Number of boundaries $p:\tilde{S} \longrightarrow S$

$$\begin{tikzcd}
\text{boundary component in } \tilde{S} \arrow[d, shift right=3, "p"'] \\
\text{boundary component in } S \arrow[u, shift right=3, "\text{lifts to one or more}"']
\end{tikzcd}$$

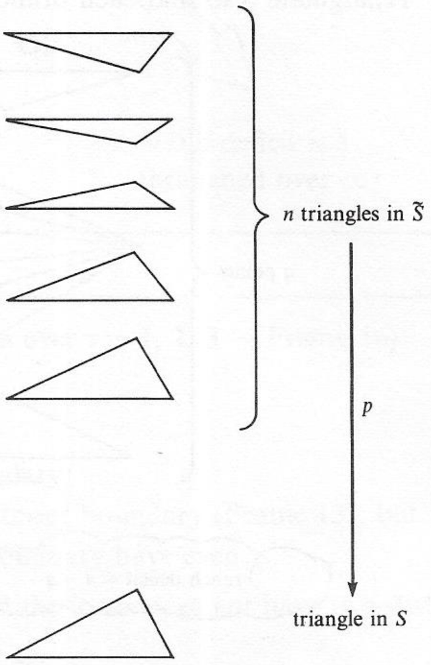
Hence we *cannot* have, say, $A^2 \xrightarrow{n} S^2$ or $K^2 \xrightarrow{n} M^2$.

FRAME 14

Theorem B Euler characteristic for an unbranched cover

Unbranched cover $\tilde{S} \xrightarrow{n} S$

$$\chi(\tilde{S}) = n\chi(S)$$



Proof

Triangulate S with

N_0 vertices,
N_1 edges,
N_2 faces.

$\chi(S) = N_0 - N_1 + N_2$.

The triangulation lifts to $\tilde{S}$ with

nN_0 vertices,
nN_1 edges,
nN_2 faces.

Hence, we *cannot* have, say, $T^2 \xrightarrow{n} S^2$, because

$$0 = \chi(T^2) \neq n\chi(S^2) = 2n.$$

Neither can we have $S^2 \xrightarrow{n} S^2$, because

$$2 = \chi(S^2) \neq n\chi(S^2) \quad \text{(unless } n = 1\text{)}.$$

In view of the above result, how do you account for the locus $w^2 = z$ (TV 5) being a *sphere* which double-covers the z-sphere?

RESTART THE TAPE

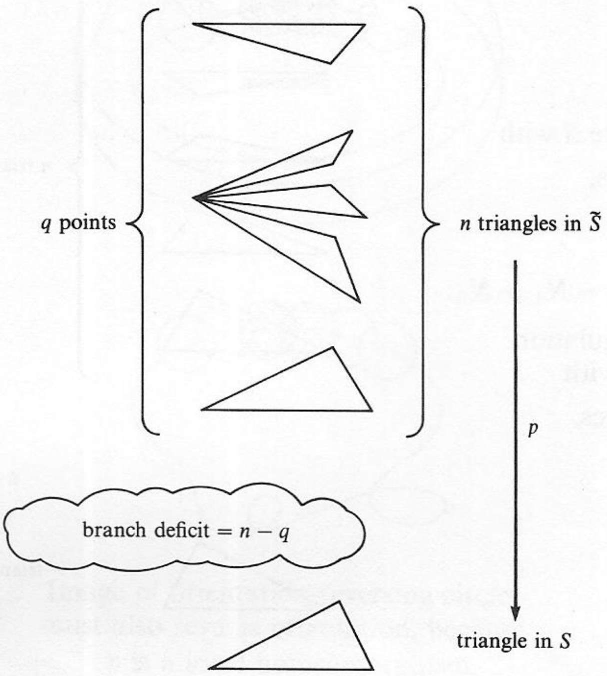
FRAME 15

Theorem B′ Euler characteristic in general

$\tilde{S} \xrightarrow{n} S$ $\qquad\delta =$ "total branch deficit"

$\chi(\tilde{S}) = n\chi(S) - \delta$ $\qquad = $ sum of branch deficits

Proof: Triangulate $\tilde{S}$ so that each branch point of $\tilde{S}$ sits over a vertex of S.



Then the triangulation of S lifts to one in $\tilde{S}$ with nN_2 faces, nN_1 edges, but $nN_0 - \delta$ vertices.

Each of the two branch points of the locus $w^2 = z$ has deficit *one*. So $\delta = 2$ and the formula works:

$$\begin{tikzcd} \text{locus } w^2 = z & z\text{-sphere} \\ S^2 \arrow[r, "2"] & S^2 \end{tikzcd}$$

$$2 = \chi(S^2) = 2\chi(S^2) - \delta = 2 \times 2 - 2$$

RESTART THE TAPE

FRAME 16

The locus $w^2 = (z-1)(z-2)(z-3)$

$$w = \pm\sqrt{(z-1)(z-2)(z-3)}$$

So the locus is a 2-fold cover of the z-sphere.
Branch points over $z = 1, 2, 3$ (see Section 5.2).

Over $z = \infty$?

FRAME 17

The locus $w^2 = (z-1)(z-2)(z-3): \tilde{S} \xrightarrow{2} S^2$

$$\chi(\tilde{S}) = 2\chi(S^2) - \delta$$
$$= 4 - (3 + \text{deficit at } \infty)$$

OR

= 1 if deficit is 0
(two points over ∞)

= 0 if deficit is 1
(branched over ∞)

FRAME 18

$\tilde{S} = T^2$ branched over ∞ (as well as over $z = 1, 2, 3$ — Frame 16)

Proof

$\chi(\tilde{S}) \neq 1$ because:

(a) S^2 is orientable without boundary;

(b) $\tilde{S}$ is orientable (Frame 12) without boundary (Frame 13); but

(c) orientable surfaces without boundary have even χ.

Hence deficit at ∞ is not zero, and the locus does *not* have two distinct points over ∞ (Frame 17).

(Neither can it be *pinched* over infinity, because, if it were, a cover with two distinct points there could be constructed by *unpinching* the locus.)

So $\chi(\tilde{S}) = 0$ (Frame 17) and:

(a) $\tilde{S}$ is branched over ∞ (Frame 17);

(b) $\tilde{S} = T^2$ (the only orientable surface without boundary which has $\chi = 0$).

THE TAPE ENDS WITH AN AMPLIFICATION OF THE ABOVE ARGUMENTS.

Post-TV Work

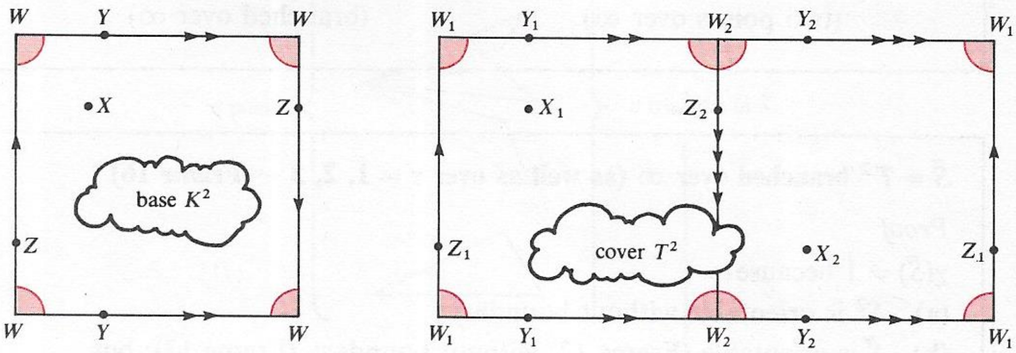
SAQ 4 Show that the map $T^2 \xrightarrow{2} K^2$ in Frame 10 is a local homeomorphism everywhere, by looking at representative points of K^2:

(a) in the interior of the polygon;

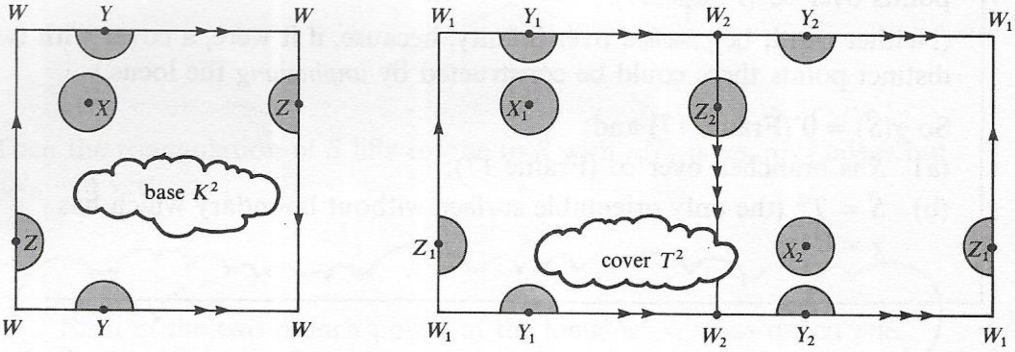
(b) on the edges;

(c) on the vertices.

SOLUTION TO SAQ 4 Look at interior point X, edge points Y and Z, vertices W:



and consider a disc surrounding each point in the base and the discs this lifts to in the cover.



■

6.3 Riemann Surfaces and Covering Surfaces of the Sphere

In TV6 you saw how to construct covering surfaces of the Möbius band M^2 by taking several copies of the polygonal representation and glueing them together. In Section 3.2, **B** describes the method more generally, and shows how the algebraic methods of *Unit 4* can be used to classify the resulting surface.

Important convention on notation

Blackett uses the symbol $\bar{S}$ to denote a covering surface. Thus he would describe a covering of the sphere by

$$p: \bar{S} \longrightarrow S^2.$$

We have chosen to use the symbol $\tilde{S}$ (which is generally accepted) in order to avoid confusion with the other interpretation of $\bar{S}$ (as the closure of S). It might be a good idea to change $\bar{S}$ to $\tilde{S}$ as you read through the next passage, but remember this convention for the *whole* of Chapter 3 of **B**.

READ B Section 3.2 from page 94 to page 96, line −7.

TES (i) *Page 95, lines 2–9*
"permute the subscripts..." Blackett insists that the edge equations you obtain in this way cannot be divided into two sets that have no edge symbols in common. This avoids covering a surface by disjoint surfaces.

(ii) *Page 95, line 17*
This sentence requires a certain amount of clarification. To obtain the second system of edge equations, **B** has applied the permutation

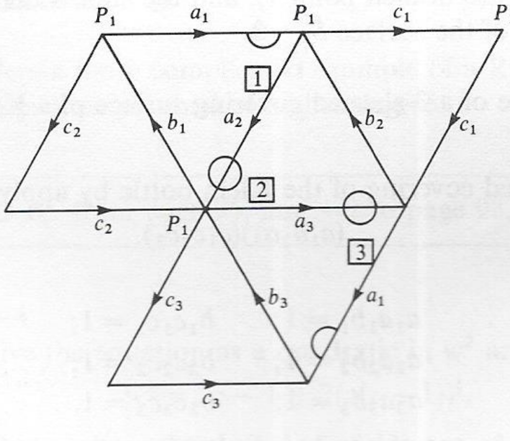
$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix} \qquad (\text{or } (a_1 a_2 a_3))$$

to the second occurrence of each a_i in the first system. This yields a 3-fold covering surface $\tilde{S}$ of the Klein bottle, which we shall analyse in SAQ 5. However, we may equally well choose the permutation

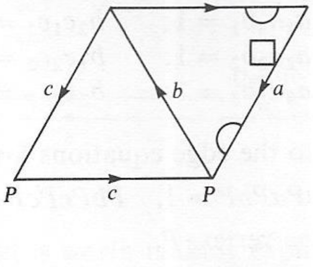
$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_1 \end{pmatrix} \qquad (\text{or } (a_1 a_2 a_3)(c_1 c_2 c_3)),$$

and this too will give a 3-fold covering surface of the Klein bottle. That these two coverings are topologically different is made clear in SAQ 6. One must approach **B**'s method with this caution in mind. Interweaving n copies of a system of edge equations by a particular permutation gives an n-fold covering surface; choosing a different permutation also gives an n-fold covering surface, which *may or may not be the same as the first one.*

(iii) *Page 95, lines 19–21*
A partial assembly of the six triangles with the given edge equations is as follows.



When we compare this with the Klein bottle (S):



it is clear that each interior point of the topological polygon defining the Klein bottle has been copied three times, as has each point on the edge (remember to watch the identifications). However, the vertex P on the Klein bottle corresponds to exactly *one* vertex upstairs, namely P_1. So P_1 is a branch point (of order $3 - 1 = 2$).

Now consider the claim that there is a local homeomorphism between $\tilde{S}$ and S, provided one avoids the vertex P_1. At interior points of the polygon for S, the argument is nearly trivial. We have depicted a little disc in S and the three little discs which cover it — you should make sure you see that the second of these is in the correct place.

For points on the edges which are not vertices, the checking is a little more difficult. We have shown the copies in $\tilde{S}$ of a disc-like neighbourhood of a point on the edge a in S. You should check that it fits.

The projection map fails to be a local homeomorphism at P_1 because *every* disc containing P_1 also contains a point (namely P_1 itself) where the map from $\tilde{S}$ to S is 3–1 rather than 1–1.

(iv) *Page 96, lines 16–18*

For reference, we repeat that some properties of S are "inherited" directly by $\tilde{S}$. The important ones are:

(a) if S is orientable, so is $\tilde{S}$;

(b) if S is closed (no boundary), so is $\tilde{S}$.

In particular, since a sphere is both closed and orientable, so are all its covering surfaces, and hence every Riemann surface is closed and orientable.

(v) *Page 96, line 24*

$\tilde{S}$ is obtained from S by making n copies and identifying some of the vertices, so if S has N_2 faces, N_1 edges and N_0 vertices, and therefore Euler characteristic

$$\chi_S = N_2 - N_1 + N_0,$$

$\tilde{S}$ has nN_2 faces and nN_1 edges. It has $nN_0 - \sum_{P_i} (k_i - 1)$ vertices, where $k_i - 1$ is the order of the branch point P_i, and the sum is taken over all the branch points of the surface $\tilde{S}$. ▲

SAQ 5 Analyse the example of a 3-sheeted covering surface of a Klein bottle on **B** page 95.

SAQ 6 Construct a 3-sheeted covering of the Klein bottle by applying the permutation

$$(a_1a_2a_3)(c_1c_2c_3)$$

to the three copies

$$\begin{aligned} a_1a_1b_1 &= 1; & b_1c_1c_1 &= 1; \\ a_2a_2b_2 &= 1; & b_2c_2c_2 &= 1; \\ a_3a_3b_3 &= 1; & b_3c_3c_3 &= 1. \end{aligned}$$

Analyse this surface and show that it is different from the example in SAQ 5.

SOLUTION TO SAQ 5 $\tilde{S}$ is given by

$$\begin{aligned} a_1a_2b_1 &= 1; & b_2c_1c_1 &= 1; \\ a_2a_3b_2 &= 1; & b_1c_2c_2 &= 1; \\ a_3a_1b_3 &= 1; & b_3c_3c_3 &= 1. \end{aligned}$$

Inserting vertex symbols into the edge equations for S,

$$PaPaPbP = 1, \quad PbPcPcP = 1,$$

we find that there is only one vertex, P.

In $\tilde{S}$, the vertices above P are all identified, say to P_1, as we saw earlier. P_1 is the only branch point, of order 2. The Klein bottle has $N_2 = 2$ faces, $N_1 = 3$ edges and $N_0 = 1$ vertex ($\chi_S = 2 - 3 + 1 = 0$), so we obtain for $\tilde{S}$

$$\chi_{\tilde{S}} = 3\chi_S - \delta = -2.$$

$\tilde{S}$ has no boundary components, since S is closed.

It remains to see if $\tilde{S}$ is orientable. Since, for example, c_1 occurs twice in the same equation, but *not* as c_1 and c_1^{-1}, $\tilde{S}$ is not orientable.

Thus $\tilde{S}$ is a sphere with four crosscaps. ■

SOLUTION TO SAQ 6 $\tilde{S}$ is given by

$$\begin{aligned} a_1a_2b_1 &= 1; & b_1c_1c_2 &= 1;\\ a_2a_3b_2 &= 1; & b_2c_2c_3 &= 1;\\ a_3a_1b_3 &= 1; & b_3c_3c_1 &= 1. \end{aligned}$$

By inserting vertex symbols, we find

$$\begin{aligned} P_1a_1Q_1a_2R_1b_1P_1 &= 1; & R_1b_1P_1c_1Q_1c_2R_1 &= 1;\\ Q_1a_2R_1a_3P_1b_2Q_1 &= 1; & P_1b_2Q_1c_2R_1c_3P_1 &= 1;\\ R_1a_3P_1a_1Q_1b_3R_1 &= 1; & Q_1b_3R_1c_3P_1c_1Q_1 &= 1; \end{aligned}$$

i.e. $\tilde{S}$ has three distinct vertices and *no* branch points.

$$\chi_{\tilde{S}} = 3\chi_S = 0.$$

As before, $\tilde{S}$ is closed; this time it is a little more tricky to show that $\tilde{S}$ is non-orientable. Look at the above system of edge equations for $\tilde{S}$, and consider the effect of reversing *any* number of equations in the first column. This will clearly leave one of the a_is repeated in the same sense. Thus $\tilde{S}$ is, indeed, non-orientable.

We can now deduce that $\tilde{S}$ is also a Klein bottle, and this choice of permutation for the symbols of $\tilde{S}$ produces another description of the 3-fold covering

$$K^2 \xrightarrow{3} K^2$$

(cf. Frame 10).

This makes clear that the covering surface depends on the particular permutation which is chosen. ■

Blackett next considers a more complicated example of a Riemann surface obtained from a complex polynomial of higher degree.

READ B from page 96, line -7 to page 98, line 3.

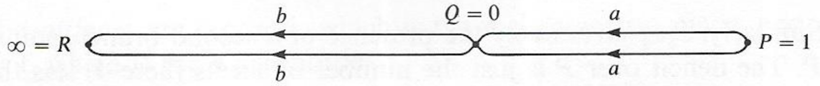
NOTES (i) *Page 96, line* -4

It is easy to solve the equation as a quadratic in w^2 and obtain

$$w^2 = 2z - 1 \pm 2(z(z-1))^{\frac{1}{2}}.$$

Now observe that each w_i^2, $i = 1, 2, 3, 4$, yields one of these expressions for w.

Blackett represents the Riemann sphere (S) like this:



The technique used next is worth making explicit.

Draw a small circle which encloses P. This circle crosses the edge a, and we know that the two possibilities for $\sqrt{z-1}$ interchange over this point (see, for example, Frame 20 of Section 5.2). As z moves round the circle, the sign of $\sqrt{z-1}$ changes when it crosses a. However, the sign of $\sqrt{z}$

does not change, and so the necessary permutation on the ws is given by

$$w_1 = \sqrt{z} + \sqrt{z-1} \leftrightarrow \sqrt{z} - \sqrt{z-1} = w_2$$

and

$$w_3 = -\sqrt{z} + \sqrt{z-1} \leftrightarrow -\sqrt{z} - \sqrt{z-1} = w_4.$$

Thus one of the necessary permutations to interweave

$$a_1b_1b_1^{-1}a_1^{-1} = 1; \quad a_2b_2b_2^{-1}a_2^{-1} = 1; \quad a_3b_3b_3^{-1}a_3^{-1} = 1; \quad a_4b_4b_4^{-1}a_4^{-1} = 1$$

is $a_1 \leftrightarrow a_2$, $a_3 \leftrightarrow a_4$ (or $(a_1a_2)(a_3a_4)$).

Now draw a circle which encloses both P and Q, which crosses edge b in the z-plane. As z crosses b, we know (again from *Unit* 5) that the two possibilities for $\sqrt{z}$ are interchanged. Since the circle also contains the point P, the values of $\sqrt{z-1}$ are also interchanged. Therefore the interweaving which takes place upstairs as z crosses b is given by

$$w_1 = \sqrt{z} + \sqrt{z-1} \leftrightarrow -\sqrt{z} - \sqrt{z-1} = w_4$$

and

$$w_2 = \sqrt{z} - \sqrt{z-1} \leftrightarrow -\sqrt{z} + \sqrt{z-1} = w_3.$$

Thus the final system of edge equations is obtained from the four copies by applying the permutations

$$(a_1a_2)(a_3a_4) \quad \textit{and} \quad (b_1b_4)(b_2b_3),$$

giving

$$a_1b_1b_4^{-1}a_2^{-1} = 1; \quad a_2b_2b_3^{-1}a_1^{-1} = 1; \quad a_3b_3b_2^{-1}a_4^{-1} = 1; \quad a_4b_4b_1^{-1}a_3^{-1} = 1.$$

(ii) *Page 97, line −11*

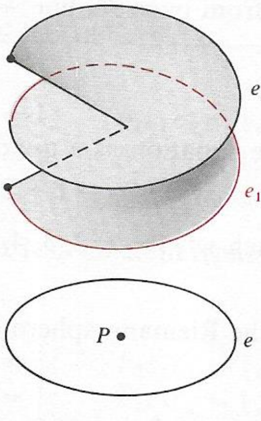
Recall that the Riemann surface is the locus with the pinch points undone.

(iii) *Page 98, line 2*

The reference should be to Example 1.

The significance of Example 2 is that it gives a neat way of spotting *two* branch points covering a given point on the Riemann sphere (we have not encountered such a thing before).

A small circle e around P lifts to two curves e_1e_2 and e_3e_4 in the 4-sheeted covering. Moreover, e_1 lies in one sheet and e_2 in another. Hence the first lifted circle overlaps two sheets, and gives a picture just like a branch point.



Similarly, e_3e_4 gives away the presence of a second branch point covering P. The deficit over P is just the number of sheets (here 4) less the number of points covering P (2), i.e. 2. Notice that it is clear from the equation in Example 1 that P is an anomalous point, for $z = 1$ gives ± 1 as the only possible values of w.

For a general polynomial in z and w, the corresponding Riemann surface can give a very complicated branched covering of the Riemann sphere. The above gives a possible method of analysing the nature of the branch points, although we shall not pursue it further in this course. ▲

SAQ 7 **B** page 107, Question 2.

SAQ 8 **B** page 107, Question 3.

SOLUTION TO SAQ 7 Inserting vertex symbols into the equation of S gives

$$PaQbRcSdTd^{-1}Sc^{-1}Rb^{-1}Qa^{-1}P = 1.$$

Now insert vertex symbols in the given system for $\tilde{S}$, obtaining

$$P_1a_1Q_1b_1R_1c_1S_1d_1T_1d_2^{-1}S_1c_1^{-1}R_1b_3^{-1}Q_1a_2^{-1}P_1 = 1;$$
$$P_1a_2Q_1b_2R_1c_2S_1d_2T_1d_3^{-1}S_2c_3^{-1}R_1b_2^{-1}Q_1a_1^{-1}P_1 = 1;$$
$$P_2a_3Q_1b_3R_1c_3S_2d_3T_1d_1^{-1}S_1c_2^{-1}R_1b_1^{-1}Q_1a_3^{-1}P_2 = 1.$$

Thus P is covered by P_1, P_2
Q by Q_1
R by R_1
S by S_1, S_2
and T by T_1.

Thus P_1, S_1 are branch points of order 1, and Q_1, R_1, T_1 are branch points of order 2.

The total deficit is therefore 8, and the Euler characteristic of $\tilde{S}$ is $3 \times 2 - 8 = -2$.

Since $\tilde{S}$ is a covering of the sphere, it is closed and orientable. Hence it is a two-holed torus. ■

SOLUTION TO SAQ 8 Inserting vertex symbols for S gives

$$PaQa^{-1}PbRb^{-1}PcSc^{-1}PdTd^{-1}P = 1;$$

and then we obtain, for $\tilde{S}$:

$$P_1a_1Q_1a_2^{-1}P_2b_1R_1b_2^{-1}P_1c_1S_1c_2^{-1}P_2d_1T_1d_2^{-1}P_1 = 1;$$
$$P_2a_2Q_1a_1^{-1}P_1b_2R_1b_1^{-1}P_2c_2S_1c_1^{-1}P_1d_2T_1d_1^{-1}P_2 = 1.$$

The only deficit occurs over Q, R, S, T, and Q_1, R_1, S_1, T_1 are all branch points of order 1. The total deficit is 4, so the Euler characteristic of $\tilde{S}$ is $2 \times 2 - 4 = 0$. Since $\tilde{S}$ covers a sphere, it is closed and orientable. Hence $\tilde{S}$ is a torus. ■

Blackett now studies unbranched double covers of non-orientable surfaces, and shows that they can be orientable. This will not surprise you, in view of our earlier example of the sphere and the real projective plane, but he proves it in general.

READ B page 98, line 4 to page 99, line -6.

NOTE *Page 98, line -8*

To make sure you understand this argument, we shall write it out for the case $p = 0$.

If $p = 0$, the surfaces are $c_1c_1 = 1$, which is the real projective plane, and $c_1c_1c_2c_2 = 1$, which is the Klein bottle.

The projective plane is double covered by $c_{11}c_{12} = 1$, $c_{12}c_{11} = 1$, which reduces to $c_{11}c_{11}^{-1} = 1$, which is the sphere.

The Klein bottle is double covered by $c_{11}c_{12}c_{21}c_{22} = 1$, $c_{12}c_{11}c_{22}c_{21} = 1$, which reduces to $c_{11}c_{21}^{-1}c_{22}^{-1}c_{11}^{-1}c_{21}c_{22} = 1$, using $c_{12} = c_{21}^{-1}c_{22}^{-1}c_{11}^{-1}$.

This in turn gives $c_{11}c_{22}^{-1}c_{21}^{-1}c_{11}^{-1}c_{21}c_{22} = 1$, using $xBCx^{-1} = xCBx^{-1}$, and finally $c_{11}c_3^{-1}c_{11}^{-1}c_3 = 1$, writing $c_{21}c_{22} = c_3$.

So this double cover of the Klein bottle is a torus. ▲

SAQ 9 B page 107, Question 6.

SOLUTION TO SAQ 9 Let $\tilde{S}$ be an unbranched covering of S, and suppose it is closed, orientable, and has genus 3. Then S must be closed, since $\tilde{S}$ has at least as many boundary curves as S. Since $\tilde{S}$ is unbranched, the total branch point order, δ, is zero and so

$$2 - 2 \times 3 = \chi_{\tilde{S}} = n\chi_S.$$

There are now only three cases to consider:

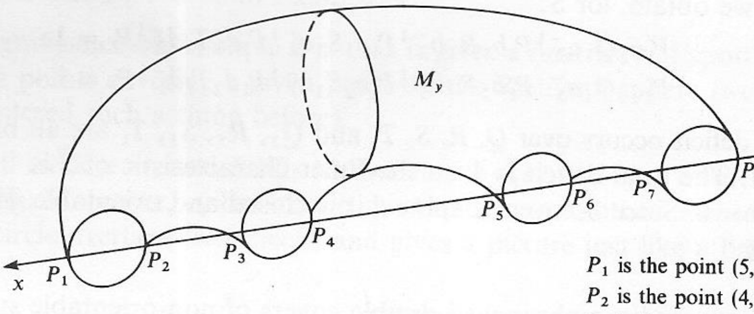
$n = 1$ simply gives $S = \tilde{S}$, which is the trivial case;
$n = 2$ gives $\chi_S = -2$;
$n = 4$ gives $\chi_S = -1$.

If $\chi_S = -1$, since S has no boundary curve, S must be non-orientable with genus 3 — i.e. a sphere with 3 crosscaps.

If $\chi_S = -2$, *either* S is orientable and therefore a two-holed torus,
or S is non-orientable and is a sphere with 4 crosscaps. ■

READ B page 99, line -5 to page 101, line 14.

NOTE S is a 3-holed torus considered as a double cover of each of S_x, S_y, S_z and S_0. We take S_x first. S is divided by the coordinate plane $y = 0$ into two parts: M_y (shown below) and its mirror image, M'_y. Mark the eight points where M_y meets the x-axis, $P_1, \ldots, P_8$. Under the identification $(x, y, z) \leftrightarrow (x, -y, -z)$, points of M_y and M'_y are paired *unless* $y = z = 0$; so $P_1, \ldots, P_8$ are the branch points of S_x, and each branch point has order 1. From the formula relating Euler characteristics,



P_1 is the point $(5, 0, 0)$
P_2 is the point $(4, 0, 0)$
P_3 is the point $(2, 0, 0)$
P_4 is the point $(1, 0, 0)$

$$\chi_S = n\chi_{S_x} - \text{(total branch point order)},$$

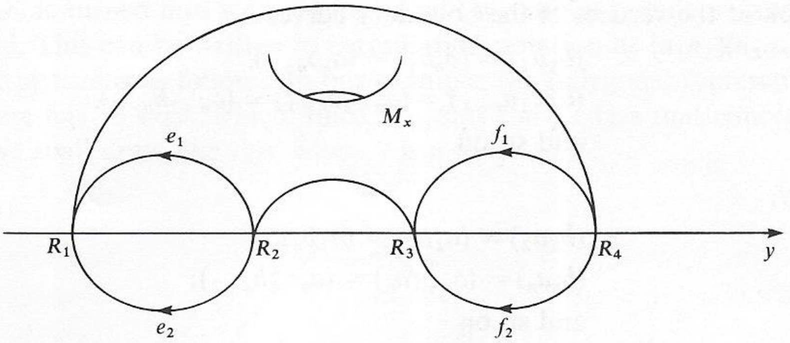
using $\chi_S = -4$, $n = 2$, we obtain $\chi_{S_x} = 2$ and S_x is a sphere.

You can literally see this by observing that $(x, 0, z)$ and $(x, 0, -z)$ are to be identified, which has the effect of joining together the two arcs P_1P_2 (and, likewise, each of the three pairs of arcs P_3P_4, P_5P_6, P_7P_8). So each circular boundary of M_y is "sealed", and M_y is transformed into a topological sphere.

Next we consider S_y. S is split into two parts by the plane $x = 0$: M_x (shown on the next page) and its mirror image, M'_x. M_x meets the y-axis at four points, R_1, R_2, R_3, R_4. Points of M_x, M'_x are paired unless $x = z = 0$. Thus R_1, R_2, R_3 and R_4 are all branch points of order 1. Then

$$\chi_{\tilde{S}} = -4 = 2\chi_S - 4$$

gives $\chi_S = 0$ (i.e. S is a torus or a Klein bottle).



Blackett's proof that S is a torus is rather long and unilluminating. However, it is geometrically obvious from the picture when we notice that the identification $(0, y, z) \leftrightarrow (0, y, -z)$ glues the pairs of arcs R_1R_2, R_3R_4 together, so as to "seal" the holes.

We could apply the same strategy to S_z. However, it is quicker to note that S_z may be obtained from M_y by identifying $(x, 0, z)$ with $(-x, 0, z)$, thus joining the inner edges and outer edges in pairs, or from M_x by identifying $(0, y, z)$ with $(0, -y, z)$, which identifies e_1 with f_1^{-1}, e_2 with f_2^{-1}. Either method produces a two-holed torus.

S_0 is produced from M_x by identifying $(0, y, z)$ with $(0, -y, -z)$, i.e. e_1 with f_2^{-1} and f_1 with e_2^{-1}. This is the identification which turns a cylinder into a Klein bottle, so it turns M_x into a Klein bottle with a handle, i.e. a sphere with four crosscaps. ▲

SAQ 10 **B** page 107, Question 7.

SAQ 11 **B** page 108, Question 8.

OLUTION O SAQ 10 Since the surface S is not orientable, its edge equation has a repeated edge symbol occurring twice in the same sense, and so looks like

$$\dots c \dots c \dots = 1.$$

A $(2n + 1)$-sheeted covering has equations which look like

$$\dots c_1 \dots c_{\sigma(1)} \dots = 1$$
$$\dots c_2 \dots c_{\sigma(2)} \dots = 1$$
$$\dots c_{2n+1} \dots c_{\sigma(2n+1)} \dots = 1,$$

where σ is a permutation of $1, 2, \dots, (2n + 1)$ (i.e. $c_{\sigma(1)}, \dots, c_{\sigma(2n+1)}$ is just $c_1, c_2, \dots, c_{2n+1}$, perhaps in a different order).

Now, the sum of all the exponents of symbols such as c_i is currently $4n + 2$. To show $\tilde{S}$ to be orientable, we must invert some equations (say r of them) to reduce this sum to zero. Inverting r equations reduces the index sum by $4r$ (try it!). Hence

$$4r = 4n + 2.$$

This is not solvable for integer r. Hence $\tilde{S}$ cannot be orientable.
(We first encountered this situation in the solution to SAQ 6.) ■

LUTION O SAQ 11 One description of a Möbius band is

$$aab = 1.$$

Make n copies of this, and use the n-cycle $(a_1 \dots a_n)$ to interweave them:

$$a_1a_2b_1 = 1$$
$$a_2a_3b_2 = 1$$
$$\vdots$$
$$a_na_1b_n = 1.$$

Now look at the vertices of the boundary curves b_i:

$$(b_1a_1) = (a_na_1) = (a_nb_{n-1}),$$
$$(b_{n-1}a_{n-1}) = (a_{n-2}a_{n-1}) = (a_{n-2}b_{n-3}),$$
and so on.

Similarly,

$$(b_2a_2) = (a_1a_2) = (a_1b_n),$$
$$(b_na_n) = (a_{n-1}a_n) = (a_{n-1}b_{n-2});$$
and so on.

If n is even, these boundary curves fall into two distinct circles:

$$b_1b_3\ldots b_{n-1},$$
$$b_2b_4\ldots b_n;$$

whereas if n is odd, *all* the boundary edges form one circle.

Thus if n is even we have a cylinder, and if n is odd a Möbius band. ■

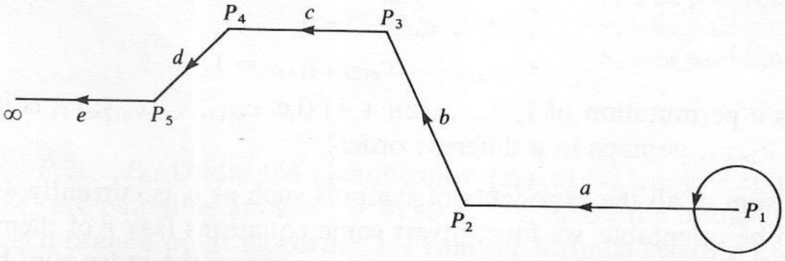
6.4 More Examples of Riemann Surfaces

This section is based on **B** Section 3.3, in which **B** describes the Riemann surfaces of some complex loci other than quadratics. We outline the method which underlies his treatment of various examples.

Given the locus

$$f(z, w) = a_0(z)w^n + \cdots + a_n(z) = 0, \qquad (*)$$

it is clear from *Unit 5* that there are generally n values of w for each value of z. In any particular case, you will need to be careful in your analysis to find where there are fewer than n values of w; these exceptional points will give rise to branch points on the Riemann surface corresponding to the locus. The first task is to find these branch points, say $P_1, \ldots, P_k$. When this is done, one draws curves P_1P_2, $P_2P_3, \ldots$, $P_k\infty$ on the Riemann sphere, and labels them a, b, c, d, $e, \ldots$, as shown.



The edge equation for this polygonal representation of the sphere is

$$abcdee^{-1}d^{-1}c^{-1}b^{-1}a^{-1} = 1.$$

Make n copies of this:

$$a_ib_ic_id_ie_ie_i^{-1}d_i^{-1}c_i^{-1}b_i^{-1}a_i^{-1} = 1,$$

thereby describing n cut spheres.

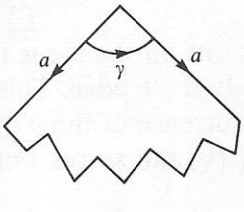
The crucial question is, how to permute the indices to get the Riemann surface, i.e. how are the edges of the different cut spheres identified?

For this we consult the equation $(*)$ which we hope to write in explicit form as

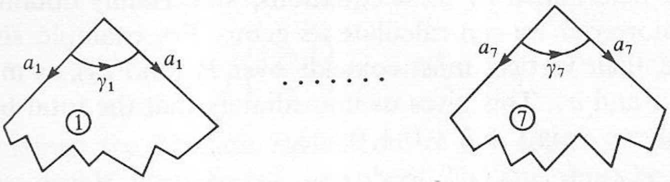
$$w = \text{some } n\text{-valued expression in } z. \qquad (\dagger)$$

Now we take z on a loop γ enclosing P_1 but no other P_i or ∞, and read off from (†) what happens to the w-values. Suppose $n = 7$, for example: it might be that w_1 is turned into w_2 as z crosses the cut a, and w_2 is turned into

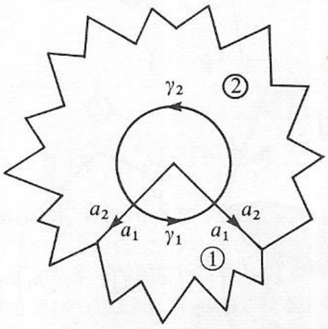
w_1; that w_7 is turned into w_3, w_3 into w_5 and w_5 into w_7, while w_4 and w_6 are unchanged. This can be written in permutation notation as $(a_1a_2)(a_3a_5a_7)$. The geometric picture is as follows. In our example, the polygonal representation of the z-sphere has 10 sides, which, since it is only the a edges that concern us at present, we shall draw like this, where γ is a loop.



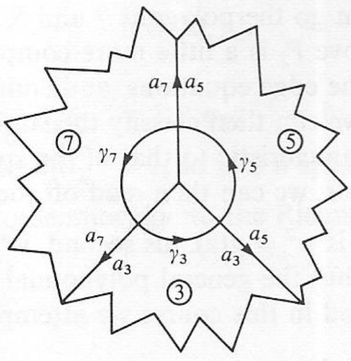
Since $n = 7$, we have 7 copies of this decagon:


a_1
γ_1
a_1
①
. . . .
a_7
γ_7
a_7
⑦

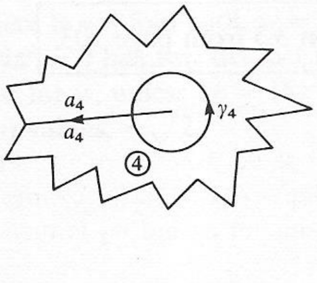
which we must fit together. The loop γ lifts to a path γ_1 connecting w_1 and w_2 values, and to a path γ_2 connecting w_2 and w_1 values, so edges a_1 and a_2 must be joined like this:

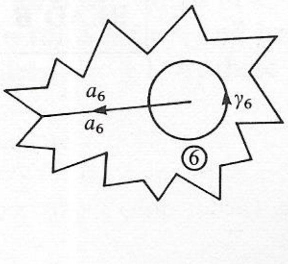

γ_2
②
a_2
a_1
γ_1
a_1
a_2
①

whereas the behaviour of the w_3, w_5 and w_7 values can be drawn like this:


a_7
a_5
⑦
γ_7
⑤
γ_5
a_7
a_3
γ_3
a_3
a_5
③

The w_4 and w_6 values are not affected as z circles p_1, so those pictures are


a_4
a_4
γ_4
④


a_6
a_6
γ_6
⑥

In the edge equation beginning with a_1, the second occurrence of a_1 is replaced by a_2, and in the equation beginning a_2, the second occurrence of a_2 is replaced by a_1. Similarly, a_3 is replaced on its second appearance by a_5, a_5 by a_7, and a_7 by a_3. So one has edge equations like

$$a_1b_1c_1d_1e_1e_{\square}^{-1}d_{\square}^{-1}c_{\square}^{-1}b_{\square}^{-1}a_2^{-1} = 1,$$

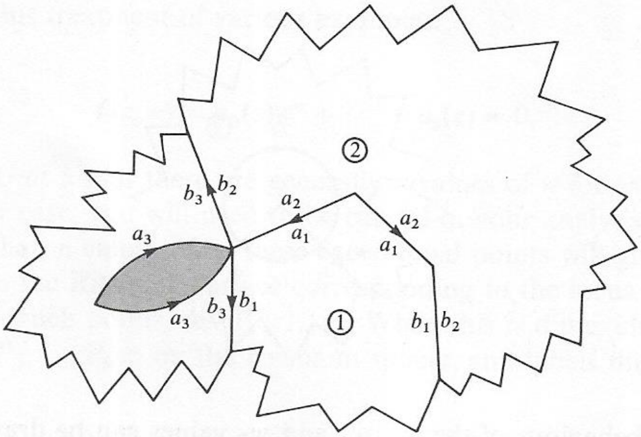
and

$$a_3b_3 \ldots\ldots\ldots\ldots\ldots\ldots b_{\square}^{-1}a_5^{-1} = 1.$$

To fill in the blanks, one traces out on the locus the lift of a z-circle δ that goes round P_1 and P_2 and crosses only the b edge. This gives a permutation of the bs which we use on the second occurrence of the b symbols in each edge equation. For the cs, we go round P_1, P_2, P_3, and so on, until we have a permutation for each symbol.

In this way, the Riemann surface is described by edge equations. Since the vertices are determined by these equations, we certainly obtain a surface in this way, and moreover we can calculate its genus. For example, since edges a_1 and a_2 coincide, their vertices must coincide over P_1 (and P_2), as must the vertices of edges a_3, a_5 and a_7. This gives us immediately that the total branch point order of points above P_1 is $1 + 2 + 0 + 0 = 3$.

The b, c and d pictures are a little more complicated. Suppose, for simplicity, that δ permutes the values of w_1, w_2 and w_3 as one crosses the b cut, but leaves the rest alone. One would write this in permutation notation as $(b_1b_2b_3)$. The picture will look something like this:



Inside the shaded region go the polygons 7 and 5. So the branch point behaviour of points above P_2 is a little more complicated. The correct procedure is to insert vertices in the edge equations, and count. Once the total branch point order is known, we can then classify the surface using the formula connecting its Euler characteristic to that of the sphere. Moreover, if we compare it with the locus, we can then read off the pinch points.

Blackett's first example is $w^2 = g(z)$, his second, $w^n = z$. These can be analysed in complete generality, but the general polynomial presents algebraic difficulties which are quite hard, and in this course we attempt only to give the flavour of the simpler examples.

READ B Section 3.3 from page 103.

EXAMPLE 5 **B** page 108, Question 1.

Solution

For each z, there are two points on the locus *except* for $z = 0, 1$. At $z = 0$, there is the single point $(0, \infty)$; at $z = 1$, the single point $(1, 0)$. You can check that at $z = \infty$ there are two points, $(\infty, 1)$ and $(\infty, -1)$, on the locus. At all other points, the two values of w are

$$w_1 = \sqrt{\left(\frac{z-1}{z}\right)}, \quad w_2 = -\sqrt{\left(\frac{z-1}{z}\right)},$$

where, as usual, we write

$$\frac{z-1}{z} = re^{i\theta} \quad \text{with } -\pi < \theta \leq \pi,$$

and define

$$\sqrt{\left(\frac{z-1}{z}\right)} = r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}.$$

In order to construct the Riemann surface out of two copies of the Riemann sphere, we must decide where to cut the sphere. The cuts must be where discontinuities in w would occur as z crosses the cut. This happens where $\dfrac{z-1}{z}$ is real and negative, because near that line we can find points with $\arg\left(\dfrac{z-1}{z}\right)$ near π and near $-\pi$. To see this, suppose $z = x + iy$ where x, y are real and y is small; then

$$\frac{z-1}{z} = 1 - \frac{1}{z} = \frac{x^2 + y^2 - x + iy}{x^2 + y^2}.$$

Assume y small compared with $|x^2 - x|$. Then if $x^2 - x$ is negative, for positive y the argument of $\dfrac{z-1}{z}$ is near π, while for negative y this is near $-\pi$. Thus we have to make a cut where $x^2 - x$ is negative, i.e. between 0 and 1. (If $x^2 - x$ is positive, then for small y the argument is near 0 so there is no sudden change in its value.)

So the surface is formed simply by interweaving two copies of the sphere cut along the edge

$$a = [0, 1],$$

and its equation is

$$a_1a_2^{-1} = 1, \quad a_2a_1^{-1} = 1.$$

The surface is equivalent to $a_1a_1^{-1} = 1$, so it is a sphere.

The locus in this case is homeomorphic to the Riemann surface.

SAQ 12 **B** page 108, Question 2.

SAQ 13 **B** page 109, Question 3.

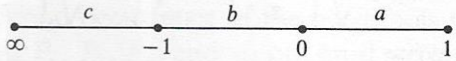
SOLUTION TO SAQ 12 We first look at the cases $z = 0$, $z = \infty$. At $z = 0$, there are three distinct values for w: $w = 0$ is a repeated root, and the other values are $w = \pm 2$.

At $z = \infty$, there is a 4-fold root, $w = \infty$. For all other values of z, $w^4 - 4w^2 + 4z^2 = 0$ has two distinct non-zero roots for w^2, and hence four distinct values for w, unless $16 = 16z^2$, i.e. unless $z = \pm 1$. At these points, w takes just two values, $\pm\sqrt{2}$.

Suppose the roots are

$$\begin{aligned} w_1 &= +\sqrt{z+1}+\sqrt{1-z}, \\ w_2 &= +\sqrt{z+1}-\sqrt{1-z}, \\ w_3 &= -\sqrt{z+1}+\sqrt{1-z}, \\ w_4 &= -\sqrt{z+1}-\sqrt{1-z}. \end{aligned}$$

Make cuts in the Riemann sphere as below.



As z crosses a from above, the argument of $z-1$ changes from near π to near $-\pi$, but the argument of $z+1$ only from small and positive to small and negative. Therefore, as we cross a, we must interchange w_1 and w_2, and also w_3 and w_4. This gives us the permutation

$$(a_1a_2)(a_3a_4).$$

The same applies as we cross b, giving

$$(b_1b_2)(b_3b_4).$$

A similar argument shows that as we cross c we have to interchange both square root signs, and hence interchange w_1 and w_4, and also w_2 and w_3. This gives us the permutation $(c_1c_4)(c_2c_3)$.

Writing the equation for the Riemann sphere with cuts a, b, c as

$$abcc^{-1}b^{-1}a^{-1}=1,$$

we see that the system for the Riemann surface of w is

$$\begin{aligned} a_1b_1c_1c_4^{-1}b_2^{-1}a_2^{-1} &= 1; \\ a_2b_2c_2c_3^{-1}b_1^{-1}a_1^{-1} &= 1; \\ a_3b_3c_3c_2^{-1}b_4^{-1}a_4^{-1} &= 1; \\ a_4b_4c_4c_1^{-1}b_3^{-1}a_3^{-1} &= 1. \end{aligned}$$

The insertion of vertices technique shows that we have four vertices above 0 and 2 vertices above each of 1, -1 and ∞. This means that the Riemann surface has three branch points, each of order 2, and so the total branch point order is 6. The Euler characteristic is therefore $4\times 2-6=2$, and the Riemann surface is again a sphere.

(We obtain the locus by identifying two points over 0, two points over ∞; so the locus is a *doubly* pinched sphere.) ■

SOLUTION TO SAQ 13 If $z=0$, then $zw^3-3zw+2z+1=0$ has no finite solution, so $(0,\infty)$ is a 3-fold repeated root.

There are no other points of the form (z,∞), $z\neq\infty$, on the locus, since, on setting $\tilde{w}=1/w$ and $\tilde{w}=0$, $z-3z\tilde{w}^2+(2z+1)\tilde{w}^3=0$ has $z=0$ as its only solution. If $z=\infty$, then set $\tilde{z}=1/z$ and study $w^3-3w+2+\tilde{z}=0$ when $\tilde{z}=0$. As $w^3-3w+2=(w-1)^2(w+2)$, $z=\infty$ is covered by $(\infty,1)$ twice, and by $(\infty,-2)$. If $z\neq 0,\infty$, consider $w^3-3w+2+1/z=0$. This has repeated roots if and only if $4(-27)+27(2+1/z)^2=0$, i.e. if and only if $z=-\frac{1}{4}$.

When $z=-\frac{1}{4}$,

$$w^3-3w+2+1/(-\tfrac{1}{4})=w^3-3w-2=(w+1)^2(w-2).$$

So the repeated root is $w=-1$.

We list our conclusions so far: the multiple points occur at

$$\begin{array}{ll} (z, w) = (0, \infty), & \text{which is a 3-fold point,} \\ (z, w) = (\infty, 1), & \text{which is a double point, and} \\ (z, w) = (-\frac{1}{4}, -1), & \text{which is a double point.} \end{array}$$

We must determine which of the points are branch points (of whatever order), and which are pinch points. Let the Riemann surface be denoted by $\tilde{S}$. It is a 3-3-fold branched covering of S. The maximum branch point orders at each branch point are 2, 1, 1, respectively, so the maximum value of δ, the total branch point order, is 4. We have

$$\chi_{\tilde{S}} = n\chi_S - \delta,$$

i.e. $$\chi_{\tilde{S}} = 3 \times 2 - \delta \geq 3 \times 2 - 4 = 2.$$

On the other hand, $\chi \leq 2$ for *any* surface, so $\chi_S = 2$, and $\tilde{S}$ is a sphere branched over 0, ∞ and $-\frac{1}{4}$. These are necessarily branch points of orders 2, 1, 1, respectively.

A Summary of Units 5 and 6

The central idea in *Unit 5* is that of the complex conic. You should be able to determine whether a given complex conic is

(i) a sphere homeomorphic to the z-sphere, or

(ii) a sphere double covering the z-sphere and branched over two points, or

(iii) a pinched sphere which double covers the z-sphere and is branched over two points.

In *Unit 6*, these ideas are generalized to include:

(a) branched and unbranched n-fold coverings of the sphere, called Riemann surfaces;

(b) branched and unbranched n-fold coverings of arbitrary surfaces;

(c) the Riemann surface associated with a complex locus. Be warned that, in general, a complex locus may have very complicated pinch points; these are all unpinched in the Riemann surface, which is genuinely a surface.

Covering surfaces can be produced:

(1) via quotient spaces using symmetries of a surface;

(2) from copies of a polygonal representation and edge equation by permuting the indices;

(3) from complex loci by the technique of geometric compactification.

Covering surfaces are analysed by the classification theorem, as follows. Given an n-fold covering of S by $\tilde{S}$ ($\tilde{S} \xrightarrow{n} S$), with total branch order δ,

(i) the Euler characteristics are related by $\chi(\tilde{S}) = n\chi(S) - \delta$;

(ii) the number of boundary components are related by $\beta(S) \leq \beta(\tilde{S}) \leq n\beta(S)$;

(iii) determine whether or not S is orientable.

In particular, these relationships also show that certain covering spaces are impossible.

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